HIGH DIMENSION PRÜFER DOMAINS OF INTEGER-VALUED POLYNOMIALS

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ABSTRACT. Let V be any valuation domain and let E be a subset of the quotient field K of V. We study the ring of integer-valued polynomials on E, that is, $\text{Int}(E, V) = \{f \in K[X] \mid f(E) \subseteq V\}$. We show that, if E is precompact, then $\text{Int}(E, V)$ has many properties similar to those of the classical ring $\text{Int}(\mathbb{Z})$. In particular, $\text{Int}(E, V)$ is dense in the ring of continuous functions $\mathcal{C}(\widehat{E}, \widehat{V})$; each finitely generated ideal of $\text{Int}(E, V)$ may be generated by two elements; and finally, $\text{Int}(E, V)$ is a Prüfer domain.

INTRODUCTION

It is well known that the classical ring of integer-valued polynomials $\text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ is a Prüfer domain. This result has been extended to the ring $\text{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}$ of integer-valued polynomials on a Dedekind domain D with finite residue fields (and quotient field K) [4, Theorem VI.1.7]. In any case, if $\text{Int}(D)$ is Prüfer, for some domain D, then D is one-dimensional and thus, the dimension of $\text{Int}(D)$ is two.

Recently, noticeable attention has been given to the study of the ring of integer-valued polynomials on a subset E of the quotient field K of a domain D, that is, $\text{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}$. If $\text{Int}(E, D)$ is a Prüfer domain, then necessarily D is Prüfer; for each maximal ideal $\mathfrak{m}$ of D, $D_{\mathfrak{m}}$ is then a valuation domain and $\text{Int}(E, D_{\mathfrak{m}})$ is also Prüfer. This is the reason why in this paper, we focus on the ring

$$\text{Int}(E, V) = \{f \in K[X] \mid f(E) \subseteq V\}$$

where E is a subset of the quotient field K of a valuation domain V.

One of the useful tools to prove that $\text{Int}(\mathbb{Z})$ is Prüfer is Dieudonné's p-adic version of the classical Stone-Weierstrass theorem [10, Theorem 4]; in particular, Mahler [16, Theorem 1] gave an explicit description of the expansion of a continuous function on the ring $\mathbb{Z}_p$ of p-adic integers in series of integer-valued polynomials on $\mathbb{Z}$. Dieudonné's result was extended by Kaplansky [14] to a compact subset E of a field K endowed with a rank-one valuation. In two recent papers [1, 17], expansions of continuous functions in series

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of integer-valued polynomials were generalized to functions on subsets but, despite their generalizations to subsets, these expansions in series concerned only the case of a rank-one discrete valuation domain with finite residue field, although Kaplansky's Stone-Weierstrass theorem holds for a rank-one (not necessarily discrete) valuation domain, whatever its residue field. Hence, we came back ourselves to this question [6] and as the Stone-Weierstrass theorem is the corner stone for many algebraic results on the classical ring $\mathrm{Int}(\mathbb{Z})$, we were then able to describe (in a survey paper with no proofs [7]) similar properties for the ring $\mathrm{Int}(E, V)$ of integer-valued polynomials on a subset E of a rank-one valuation domain V (provided the completion of E is compact). Yet, we thus always obtained (new) examples of two-dimensional Prüfer domains.

In fact, we can now prove, in this paper, that Kaplansky's result can be generalized to any valuation domain, without any restriction on its dimension. A valuation field K is naturally endowed with a topology which, in, fact derives from a Hausdorff uniform structure, and thus can be completed. In the first section we extend the Stone-Weierstrass theorem in the case where E is a *precompact* subset of K, that is, such that the completion $\widehat{E}$ of E (closure of E in the completion $\widehat{K}$ of K) is compact. We mainly consider infinite subsets E (as much is already known on integer-valued polynomials on a finite subset [15]); in particular, we observe that, if such an infinite precompact subset E of K does exist, then K is in fact metrizable, and hence, compactness is equivalent to sequential compactness, which makes proofs easier.

Always under the assumption that E is precompact, in the second section we determine the prime spectrum of $\mathrm{Int}(E, V)$. Then, in the third section, using the value function associated to an ideal, we prove that each unitary finitely generated ideal of $\mathrm{Int}(E, V)$ is characterized by its values (*the almost strong Skolem property*), and that each finitely generated ideal may be generated by two elements (*the two-generators property*). We then may easily conclude in the last section that $\mathrm{Int}(E, V)$ is a Prüfer domain (that can be of arbitrary dimension).

Hypothesis and Notation. Throughout, V denotes the ring of a valuation v on a field K, with maximal ideal $\mathfrak{m}$ and value group Γ. We let E be an infinite subset of K. We denote respectively by $\widehat{V}$, $\widehat{K}$, $\widehat{\mathfrak{m}}$, and $\widehat{E}$ the completions of V, $K, \mathfrak{m}$ and E (but we simply denote by v the extension of the valuation to $\widehat{K}$).

1. Topological properties and the Stone-Weierstrass theorem

Metrizable valued field. The non-zero (fractional) principal ideals of V are the ideals of the type

$$I_\alpha = \{x \in K \mid v(x) \geq \alpha\}, \text{ for } \alpha \in \Gamma.$$

With this notation, let us recall a few topological facts, as in a standard reference such as Bourbaki [2, 3] :

There exists a unique topology compatible with the additive group structure of K for which the ideals I_α form a fundamental system of neighborhoods of 0. This topology is compatible with the field structure of K (hence, in particular, with the ring structure of V) and, if Γ is endowed with the discrete topology, the map $v : K^* \to \Gamma$ is continuous [3, VI §5, Prop. 1]. Every non-zero fractional ideal is a clopen neighborhood of (0). To this topology corresponds an additive uniform structure: a fundamental system of entourages is given by the sets formed by the pairs (x, y) such that $(x - y) \in I_\alpha, \alpha \in \Gamma$ [2, III §3, Def.1]. By the consideration of Cauchy filters, one can then define the completion $\widehat{K}$ of K. Then $\widehat{K}$ is itself a valued field, the valuation v extends to $\widehat{K}$ with the same value group Γ, the corresponding valuation ring is the completion $\widehat{V}$ of V (that is, the topological closure of V in $\widehat{K}$) and the maximal ideal of $\widehat{V}$ is the completion $\widehat{\mathfrak{m}}$ of the maximal ideal $\mathfrak{m}$ of V [3, VI §5, Prop. 5].

This topology (and correspondingly this uniform structure) is not necessarily metrizable. In fact, K is metrizable if and only if there exists a countable fundamental system of neighborhoods of (0) [2, IX §3, Prop. 1]. Thus we observe that K is metrizable if and only if there exists a sequence $\{\alpha_n\}$ in Γ such that, for each $\alpha \in \Gamma$, there exists an integer n such that $\alpha_n \geq \alpha$, or equivalently, there exists a sequence $\{I_n\}$ of (principal) fractional ideals of V such that $\bigcap_{n\in\mathbb{N}} I_n = (0)$ (letting $I_n = \{x \in V \mid v(x) \geq \alpha_n\}$). When this is the case, we can define a numerical function w on K by

$$w(x) = \max\{n \mid v(x) \geq \alpha_n\}$$

(with the convention $w(0) = \infty$). We thus explicitly obtain a metric d on K, letting

$$d(x, y) = e^{-w(x-y)}.$$

A fundamental system of entourages (for the corresponding uniform structure) is given by the sets formed by the pairs (x, y) such that $(x - y) \in I_n, n \in \mathbb{N}$. When K is metrizable, the completion can be defined in terms of Cauchy sequences. Note that, for K to be metrizable, it is sufficient that the valuation v be of finite or countable rank, or that there exists an height-one prime ideal in the valuation domain V. In fact, we shall see next that, for our concern, we can always restrict to the metrizable case.

Precompact subsets. Recall that a topological space X is said to be *precompact* if its completion is compact [2, II §4, Def. 2]. Recall also that, when X is metrizable, then X is compact if and only if every sequence $\{x_n\}$ has a cluster point in X [2, IX §2, Prop. 15], this condition being anyway necessary, even in the case where X is not metrizable. We are interested in the infinite subsets E of K which are precompact (that is, such that the topological closure $\widehat{E}$ of E in $\widehat{K}$ is compact). This allows us to restrict to the metrizable case:

Lemma 1.1. *Suppose there exists a precompact infinite subset E of K. Then K is metrizable.*

Proof. Let $\{x_n\}$ be a sequence in E, the elements of which are distinct. Assuming E to be precompact, this sequence has a cluster point x in $\widehat{E}$. In particular, for each α in the value group Γ, there is an element x_n of the given sequence, distinct from x, such that $v(x_n - x) \geq \alpha$ (that is, $(x_n - x)$ belongs to the neighborhood I_α of 0). Letting $\alpha_n = v(x_n - x)$, we clearly obtain a sequence $\{\alpha_n\}$ such that, for each α in Γ, $\alpha_n \geq \alpha$ for some n. □

A metrizable topological set is precompact if and only if, for every $\varepsilon > 0$, there exists a finite covering of X by subsets of diameter less than ε [2, IX §2, Prop. 14]. We derive the following characterization of the precompact subsets E of K (generalizing from the case of a discrete valuation domain [5, Lemma 4.4]):

Proposition 1.2. *Let E be a subset of a valued field K whose topology is metrizable, that is, there exists a sequence $\{\alpha_n\}$ in Γ, such that, for each $\alpha \in \Gamma$, there exists an integer n such that $\alpha_n \geq \alpha$. The following assertions are equivalent.*

(i) *E is precompact.*
(ii) *For each non-zero fractional ideal I of K, E meets finitely many cosets of K modulo I.*
(iii) *For each positive integer n, E meets finitely many cosets of K modulo $I_n = \{x \in K \mid v(x) \geq \alpha_n\}$.*

Example 1.3. The subset $E = \{x_n\}$ formed by a Cauchy sequence in K is a precompact subset of K.

Corollary 1.4. *Let E be a precompact subset of K. Then $v(x)$ reaches a minimum on E.*

One could say that the function v from K^* to Γ endowed with the discrete topology is continuous. More directly, if $v(y) = \alpha$, for some element $y \in E$, since E meets finitely cosets modulo I_α, it follows that, for $x \in E$, $v(x)$ takes only finitely many distinct values less than α and thus clearly reaches a minimum.

In particular, a precompact subset is a *fractional subset* of V, that is, there is a non-zero d in V such that $dE \subset V$. Replacing E by dE (which is homeomorphic to E and such that $\mathrm{Int}(dE, V)$ is isomorphic to $\mathrm{Int}(E, V)$ as a V-algebra) we can restrict ourselves to precompact subsets of V.

Continuity. If f is a polynomial with coefficients in K, and d a common denominator of its coefficients, then, for each $a, b \in V$, we have

$$v\big(f(b) - f(a)\big) \geq v(b - a) - v(d).$$

In particular, f is uniformly continuous on each subset E of V. This generalizes easily to a fractional subset:

Proposition 1.5. *Let E be a fractional subset of K and f be a polynomial with coefficients in K. Then f is a uniformly continuous map from E to K. In particular, if $f \in \mathrm{Int}(E,V)$, then f is a uniformly continuous map from E to V.*

As the image of a precompact set by a continuous function is precompact [2, II §4, Prop. 2], we obtain the following:

Corollary 1.6. *Let E be a precompact subset of K and f be a polynomial with coefficients in K. Then $f(E)$ is a precompact subset of K and $v\big(f(x)\big)$ reaches a minimum for $x \in E$.*

Localization. In general, as for any domain D, if E is a subset of K, and S is a multiplicative subset of V, we clearly have the containment

$$S^{-1}\mathrm{Int}(E,V) \subseteq \mathrm{Int}(E,S^{-1}V).$$

In the case of a precompact subset, we next show, from the previous results, that we have an equality. As V is a valuation domain, we first note that $S^{-1}V = V_{\mathfrak{p}}$ is the localization of V with respect to some prime ideal $\mathfrak{p}$.

Proposition 1.7. *Let E be a precompact subset of K, and $\mathfrak{p}$ be a prime ideal of V, then*

$$\mathrm{Int}(E,V)_{\mathfrak{p}} = \mathrm{Int}(E,V_{\mathfrak{p}}).$$

Proof. Let $f \in \mathrm{Int}(E,V_{\mathfrak{p}})$. It follows from Corollary 1.6 that $v\big(f(x)\big)$ reaches a minimum on E, say for $x = x_0$. Let $s = \big(f(x_0)\big)^{-1}$. As $f(x_0) \in V_{\mathfrak{p}}$, we have $s \notin \mathfrak{p}$, and from the definition of x_0, it follows that, for each $x \in E$, we have $v\big(sf(x)\big) \geq 0$, that is $sf \in \mathrm{Int}(E,V)$. In conclusion, f belongs to $\mathrm{Int}(E,V)_{\mathfrak{p}}$. □

Remark 1.8. Let $\mathfrak{p}$ be a prime ideal of V. Then each ideal of $V_{\mathfrak{p}}$ is also an ideal of V (since in particular, the maximal ideal of $V_{\mathfrak{p}}$ is $\mathfrak{p}$, which is contained in V) and each ideal of V contains an ideal of $V_{\mathfrak{p}}$. Thus the topologies of K defined by the valuations corresponding to V and to $V_{\mathfrak{p}}$ are the same. It follows that, if E is a precompact subset of V, it is also precompact when considered as a subset of $V_{\mathfrak{p}}$ (in fact, one can also easily see that, for each ideal I of $V_{\mathfrak{p}}$, E meets finitely many cosets modulo I).

The Stone-Weierstrass theorem. Every integer-valued polynomial on a fractional subset E of V is a uniformly continuous function from E to V. In particular, $\mathrm{Int}(E,V)$ is contained in the ring $\mathcal{C}(\widehat{E},\widehat{V})$ of continuous functions from $\widehat{E}$ to $\widehat{V}$. If E is precompact, we can establish an analogue of the Stone-Weierstrass theorem. We first show that the polynomials in $\mathrm{Int}(E,V)$ separate the points of E (a property similar to that of *interpolation domains* [8]):

Lemma 1.9. *Let E be a precompact subset of K. Then, for each $a,b \in E$, $a \neq b$, there exists $f \in \mathrm{Int}(E,V)$ such that $f(a) = 1$ and $f(b) = 0$.*

Proof. Let $\gamma = v(a-b)$. Since E is precompact, it meets finitely many cosets modulo the ideal $I_\gamma = \{x \in V \mid v(x) \geq \gamma\}$, one of them containing a (and b). We denote by $b_1, \ldots, b_n$, a system of representatives for these cosets, thus, letting $B_j = E \cap (b_j + I_\gamma)$, we have the equality $E = \bigcup_{1 \leq j \leq n} B_j$. With no loss of generality, we set $b_1 = b$, and letting $\delta_i = v(a - b_i)$, we order $b_2, \ldots b_n$ in such a way that $\gamma = \delta_1 > \delta_2 \geq \ldots \geq \delta_n$.

We prove, by induction on n, that there is a polynomial g, with coefficients in K, such that $g(b) = 0$ and $v\big(g(x)\big) \geq v\big(g(a)\big)$ for $x \in E$. Hence the lemma is proved with $f(X) = g(X)/g(a)$.

— If $n = 1$, for each $x \in E$ we have $v(x-b) \geq v(a-b)$. We can thus take $g = X - b$.

— Assume the result to be true for $n-1$: letting $E_1 = \bigcup_{1 \leq j \leq n-1} B_j$, there is a polynomial g_1 such that $g_1(b) = 0$ and $v\big(g_1(x)\big) \geq v\big(g_1(a)\big)$ for $x \in E_1$. Multiplying g_1 by a constant, we may assume that $g_1 \in \mathrm{Int}(E, V)$, and hence, that $v\big(g_1(x)\big) \geq 0$, for each $x \in E$. We let $\alpha = v\big(g_1(a)\big)$, and set $\beta = \alpha + \delta_n$. As E is precompact, so is B_n (which is contained in E) and thus, B_n meets finitely many cosets modulo $I_\beta = \{x \in V \mid v(x) \geq \beta\}$, with representatives $y_1, \ldots, y_m$. We claim that the polynomial we are looking for is

$$g = g_1 \prod_{1 \leq i \leq m} (X - y_i).$$

On the one hand, we have

$$v\big(g(a)\big) = \alpha + \sum_{1 \leq i \leq m} v(a - y_i).$$

Now, for each $y \in B_n$, writing $a - y = (a - b_n) + (b_n - y)$, with $v(b_n - y) \geq \gamma$ and $v(a - b_n) = \delta_n$, we have $v(a-y) = \delta_n$, and thus

$$v\big(g(a)\big) = \alpha + m\delta_n.$$

On the other hand, for $x \in E$, we consider two cases:

• $x \in E_1$. It follows from the choice of g_1, that we have $v\big(g_1(x)\big) \geq \alpha$. Hence, we have

$$v\big(g(x)\big) \geq \alpha + \sum_{1 \leq i \leq m} v(x - y_i).$$

Now x belongs to some B_j, $1 \leq j \leq n-1$, and for $y \in B_n$, we write

$$x - y = (x - b_j) + (b_j - a) + (a - b_n) + (b_n - y),$$

with $v(x - b_j) \geq \gamma$, $v(b_j - a) = \delta_j$, $v(a - b_n) = \delta_n$, and $v(b_n - y) \geq \gamma$. Hence $v(x-y) \geq \delta_n$. It follows that we have

$$v\big(g(x)\big) \geq \alpha + m\delta_n = v\big(g(a)\big).$$

• $x \in B_n$. As we assumed that g_1 belongs to $\mathrm{Int}(E, V)$, we have $v\big(g_1(x)\big) \geq 0$, and hence,

$$v\big(g(x)\big) \geq \sum_{1 \leq i \leq m} v(x - y_i).$$

For each i, y_i belongs to B_n, that is, y_i is in the class of x modulo I_γ, moreover one of them is in the class of x modulo I_β. We thus have

$$v\big(g(x)\big) \geq (m-1)\gamma + \beta \geq \alpha + m\delta_n = v\big(g(a)\big).$$

□

We conclude with an analogue of the Stone-Weierstrass theorem.

Theorem 1.10. *Let E be a precompact subset of K. Then $\mathrm{Int}(E,V)$ is dense in $\mathcal{C}(\widehat{E},\widehat{V})$ (for the uniform convergence topology).*

Proof. One can reduce to the case where V is complete (and hence, E is compact). Indeed, assume each $\phi \in \mathcal{C}(\widehat{E},\widehat{V})$ can arbitrarily be approximated by a polynomial (with coefficients in $\widehat{K}$). As $K[X]$ is dense in $\widehat{K}[X]$, then ϕ can be approximated by some $f \in K[X]$. But now, if $v\big(f(x)-\phi(x)\big) \geq 0$ for each $x \in \widehat{E}$, then clearly $f \in \mathrm{Int}(E,V)$.

As each continuous function can be arbitrarily approximated by a locally constant function, it is enough to prove that the characteristic function ϕ of a clopen set U can be approximated by an integer-valued polynomial: for each $A > 0$, there is $g \in \mathrm{Int}(E,V)$ such that $v\big(g(x)-1\big) \geq A$, for $x \in U$ and $v\big(g(x)\big) \geq A$, for $x \in E \setminus U$.

Choose $a \in U$. For each $b \notin U$, by the previous lemma, there is a polynomial $f_b \in \mathrm{Int}(E,V)$, such that $f_b(a) = 1$, and $f_b(b) = 0$. Since f_b is continuous, we have $v\big(f_b(x)\big) \geq A$ for x in some neighborhood of b. Since U is a clopen subset of E, U is compact, hence there is a product g_a of a finite number of such polynomials such that $g_a(a) = 1$, and $v\big(g_a(x)\big) \geq A$ for $x \in E \setminus U$. Since g_a is continuous, we have $v\big(g_a(x)-1\big) \geq A$, for x in some neighborhood of a. Since U is compact again, there is a finite product $\prod(1-g_a) = 1-g$ (with such polynomials g_a) such that $v\big(g(x)-1\big) \geq A$, for $x \in U$ and $v\big(g(x)\big) \geq A$, for $x \in E \setminus U$. □

As a locally constant function is continuous, it is easy to derive the following application, similarly to the case of $\mathrm{Int}(V)$, where V is a discrete valuation domain with finite residue field [4, Corollary III.3.5]:

Corollary 1.11. *Let E be a precompact subset of K, $U_1,\ldots,U_r$ be disjoint open subsets covering $\widehat{E}$, and $\alpha_1,\ldots,\alpha_r$ be arbitrarily chosen in the value group Γ. Then there exists a polynomial $h \in K[X]$ such that $v\big(h(x)\big) = \alpha_i$, for $x \in U_i \cap E$, $1 \leq i \leq r$.*

Proof. If $\alpha_1,\ldots,\alpha_r$ are positive, this amounts to approximating a locally constant function by an integer-valued polynomial. In the general case, add a constant $\gamma = v(d)$ to each α_i in order to make them positive, obtain an integer-valued polynomial and then divide its coefficients by d. □

2. Prime spectrum

Unitary prime ideals. We now determine the prime ideals of $\mathrm{Int}(E, V)$. Recall that an ideal of $\mathrm{Int}(E, V)$ is said to be *unitary* if it contains non-zero constants. In particular, the unitary prime ideals of $\mathrm{Int}(E, V)$ are above non-zero prime ideals of V. If $\mathfrak{p}$ is such a non-zero prime ideal of V, we denote by $\widehat{\mathfrak{p}}$ its completion. For each $\alpha \in \widehat{E}$,

$$\mathfrak{P}_{\mathfrak{p},\alpha} = \{f \in \mathrm{Int}(E, V) \mid f(\alpha) \in \widehat{\mathfrak{p}}\}$$

is clearly a prime ideal of $\mathrm{Int}(E, V)$ above $\mathfrak{p}$ and $\mathrm{Int}(E, V)/\mathfrak{P}_{\mathfrak{p},\alpha} \simeq V/\mathfrak{p}$. Assuming E to be precompact, we conversely show that all prime ideals above $\mathfrak{p}$ are of that type (similarly to the prime ideals above the maximal ideal in the case of a discrete valuation domain with finite residue field [4, Proposition V.2.3 & Theorem V.2.10]).

We denote by $\mathrm{Int}(E, \mathfrak{p})$ the polynomials in $\mathrm{Int}(E, V)$ with values in $\mathfrak{p}$.

Lemma 2.1. *Let E be a precompact subset of K and $\mathfrak{p}$ be a non-zero prime ideal of V. Then each prime ideal $\mathfrak{P}$ of $\mathrm{Int}(E, V)$ above $\mathfrak{p}$ contains the ideal $\mathrm{Int}(E, \mathfrak{p}) = \{f \in \mathrm{Int}(E, V) \mid f(E) \subseteq \mathfrak{p}\}$.*

Proof. Let $f \in \mathrm{Int}(E, \mathfrak{p})$. As $v\big(f(x)\big)$ reaches a minimum on E [Corollary 1.6], say at x_0, and as $\alpha = f(x_0)$ belongs to $\mathfrak{p}$, then $f = \alpha(f/\alpha)$ belongs to $\mathfrak{P}$. □

Theorem 2.2. *Let E be a precompact subset of K and $\mathfrak{p}$ be a non-zero prime ideal of V. Then the prime ideals of $\mathrm{Int}(E, V)$ above $\mathfrak{p}$ are in one-to-one correspondence with the elements of the completion $\widehat{E}$ of E: to each $\alpha \in \widehat{E}$ corresponds the prime ideal*

$$\mathfrak{P}_{\mathfrak{p},\alpha} = \{f \in \mathrm{Int}(E, V) \mid f(\alpha) \in \widehat{\mathfrak{p}}\}.$$

Proof. Let $\mathfrak{P}$ be a prime ideal above $\mathfrak{p}$, we first show that we have $\mathfrak{P} \subseteq \mathfrak{P}_{\mathfrak{p},\alpha}$ for some α. By way of contradiction, assume that, for each $\alpha \in \widehat{E}$, there is $f \in \mathfrak{P}$ such that $f(\alpha) \notin \widehat{\mathfrak{p}}$. Since $\widehat{E}$ is compact, it may be covered by finitely many open sets $U_i, 1 \le i \le r$, corresponding to polynomials $f_i \in \mathfrak{P}$, and elements $u_i \in V$, $u_i \notin \mathfrak{p}$, such that $f_i(x) \equiv u_i \pmod{\widehat{\mathfrak{p}}}$ for $x \in U_i$. Let $u = \prod_{i=1}^r u_i$ and consider the polynomial $g = u - \prod_{i=1}^r (u_i - f_i)$. Developing the product $\prod_{i=1}^r (u_i - f_i)$, one can see that g belongs to $\mathfrak{P}$. On the other hand, for each $x \in E$, we have $g(x) \equiv u \pmod{\mathfrak{p}}$, and hence, $(g - u) \in \mathrm{Int}(E, \mathfrak{p})$. From the previous lemma it follows that we have $(g - u) \in \mathfrak{P}$, and hence $u \in \mathfrak{P}$, reaching a contradiction.

Now, let $f \in \mathfrak{P}_{\mathfrak{p},\alpha}$. As $f(E)$ is precompact, it meets only finitely many classes modulo $\mathfrak{p}$, say with representatives $u_1, \ldots, u_r$. Thus $\prod_{i=1}^r (f - u_i)$ belongs to $\mathrm{Int}(E, \mathfrak{p})$, and hence, to $\mathfrak{P}$. Therefore, $(f - u_i)$ belongs to $\mathfrak{P}$ for some i. A fortiori, we have $(f - u_i) \in \mathfrak{P}_{\mathfrak{p},\alpha}$, and hence, $u_i \in \mathfrak{p}$, and thus finally, $f \in \mathfrak{P}$. We conclude that we have $\mathfrak{P} = \mathfrak{P}_{\mathfrak{p},\alpha}$.

Lastly, it follows from the Stone-Weierstrass theorem, that $\mathfrak{P}_{\mathfrak{p},\alpha}$ and $\mathfrak{P}_{\mathfrak{p},\beta}$ are distinct for $\alpha \neq \beta$.

□

Trivial fiber. Generalizing the case of $\mathrm{Int}(V)$ (for a discrete valuation domain with finite residue field) [4, Proposition V.2.5], we have the following, using Corollary 1.11:

Proposition 2.3. *Let E be a precompact subset of K.*

(1) *The prime ideals of* $\mathrm{Int}(E,V)$ *above* (0) *are in one-to-one correspondence with the monic irreducible polynomials of* $K[X]$*: to the irreducible polynomial* q *corresponds the prime ideal*
$$\mathfrak{P}_q = qK[X] \cap \mathrm{Int}(E,V).$$
(2) *For each non-zero prime ideal* $\mathfrak{p}$ *of* V *and each* $\alpha \in \widehat{E}$*, the ideal* $\mathfrak{P}_q$ *is contained in the ideal* $\mathfrak{P}_{\mathfrak{p},\alpha} = \{f \in \mathrm{Int}(E,V) \mid f(\alpha) \in \widehat{\mathfrak{p}}\}$ *if and only if* $q(\alpha) = 0$.

Proof. 1. As E is a fractional subset, we have $V[dX] \subseteq \mathrm{Int}(E,V)$, for some non-zero $d \in V$, and hence, $S^{-1}\mathrm{Int}(E,V) = K[X]$ where S denotes the set of non-zero elements of V.
2. Clearly, if $q(\alpha) = 0$, then $\mathfrak{P}_q \subset \mathfrak{P}_{\mathfrak{p},\alpha}$. Conversely, assume that $q(\alpha) \neq 0$ and let $\gamma = v\big(q(\alpha)\big)$. By continuity, there is a clopen neighborhood U of α in $\widehat{E}$ such that $v\big(q(x)\big) = \gamma$ for $x \in U$. It follows from Corollary 1.11 that there exists a polynomial $h \in K[X]$ such that $v\big(h(x)\big) = -\gamma$ if $x \in U$, and $v\big(h(x)\big) \geq 0$ otherwise. Clearly hq is integer-valued, $hq \in \mathfrak{P}_q$, and $hq(\alpha) \notin \widehat{\mathfrak{p}}$, thus $hq \notin \mathfrak{P}_{\mathfrak{p},\alpha}$. □

Corollary 2.4. *Let E be a precompact subset of K. The maximal ideals of* $\mathrm{Int}(E,V)$ *are the ideals* $\mathfrak{P}_{\mathfrak{m},\alpha} = \{f \in \mathrm{Int}(E,V) \mid f(\alpha) \in \widehat{\mathfrak{m}}\}$*, where* $\mathfrak{m}$ *is maximal in* V *and* $\alpha \in \widehat{E}$*, and the ideals* $\mathfrak{P}_q = qK[X] \cap \mathrm{Int}(E,V)$*, where* q *is irreducible in* $K[X]$ *with no root in* $\widehat{E}$.

Krull dimension. We denote by $\dim(R)$ the Krull dimension (finite or infinite) of a ring R.

Proposition 2.5. *Let E be a precompact subset of K. Then we have*
$$\dim(\mathrm{Int}(E,V)) = \dim(V) + 1.$$

Proof. Let α be an element of E. If $\mathfrak{p}$ and $\mathfrak{q}$ are two non-zero prime ideals of V such that $\mathfrak{p} \subset \mathfrak{q}$, we clearly have the containments
$$(0) \subset (X-\alpha)K[X] \cap \mathrm{Int}(E,V) \subset \mathfrak{P}_{\mathfrak{p},\alpha} \subset \mathfrak{P}_{\mathfrak{q},\alpha}.$$
We thus have $\dim(\mathrm{Int}(E,V)) \geq \dim(V) + 1$. The equality follows from the fact that the prime ideals of $\mathrm{Int}(E,V)$ above a non-zero prime ideal are incomparable. □

3. VALUE FUNCTIONS AND SKOLEM PROPERTIES

Value functions. Let $\mathfrak{A}$ be an ideal of $\mathrm{Int}(E,V)$. For each $\alpha \in E$, we let
$$\mathfrak{A}(\alpha) = \{f(\alpha) \mid f \in \mathfrak{A}\},$$
then $\mathfrak{A}(\alpha)$ is clearly an ideal of V, it is called the *ideal of values* of $\mathfrak{A}$ at α. If $\mathfrak{A}(\alpha)$ is finitely generated (thus in fact, principal), it is characterized by the valuation γ of a generator and we have

(1) $$\gamma = \inf\{v\big(f(\alpha)\big) \mid f \in \mathfrak{A}\}.$$

However, as we do not suppose the valuation v to be discrete, the ideals of values are not necessarily finitely generated. Nevertheless, the ideals of V are totally ordered by inclusion: to be consistent with the order on the value group Γ, we write $I \leq J$, if the ideal I contains the ideal J (in particular, the ideal (0), which corresponds to the valuation $+\infty$, is the "largest" ideal). We may thus consider the set of ideals of V as a totally ordered set Γ^*, containing Γ^+ (the set of positive elements of Γ) as a (totally ordered) subset. Moreover this ordered set Γ^* is a complete lattice: every collection of ideals has a lower bound (the union of the ideals in this collection) and an upper bound (the intersection of the ideals in this collection). Considering $\mathfrak{A}(\alpha)$ as an element of the ordered set Γ^*, we thus generalize (1) as follows:

$$\mathfrak{A}(\alpha) = \inf\{v\big(f(\alpha)\big) \mid f \in \mathfrak{A}\}. \tag{2}$$

Next, we define a function $\Psi_{\mathfrak{A}}$, from E to Γ^*, by letting

$$\Psi_{\mathfrak{A}}(\alpha) = \mathfrak{A}(\alpha) = \inf\{v\big(f(\alpha)\big) \mid f \in \mathfrak{A}\}.$$

In particular, if f is an integer-valued polynomial, we denote by Ψ_f the function corresponding to the ideal generated by f. As the corresponding ideal of values is principal, generated by $f(\alpha)$, we may consider this particular function as taking its values in $\Gamma \cup \{+\infty\}$, with $\Psi_f(\alpha) = v\big(f(\alpha)\big)$. For each ideal $\mathfrak{A}$ in $\mathrm{Int}(E, V)$, we then have

$$\Psi_{\mathfrak{A}} = \inf\Big\{\Psi_f \mid f \in \mathfrak{A}\Big\}. \tag{3}$$

In fact, as each $f \in \mathrm{Int}(E, V)$ can be considered as a (uniformly continuous) map from $\widehat{E}$ to $\widehat{V}$ [Proposition 1.5], we can extend Ψ_f as a function from $\widehat{E}$ to Γ, and similarly, each function $\Psi_{\mathfrak{A}}$ as a function from $\widehat{E}$ to Γ^*, using (2):

$$\Psi_{\mathfrak{A}} : \alpha \in \widehat{E} \to \inf\{v\big(f(\alpha)\big) \mid f \in \mathfrak{A}\}.$$

We say that $\Psi_{\mathfrak{A}}$ is the *value function* of $\mathfrak{A}$.

Remarks 3.1. 1) In general, for $\alpha \in \widehat{E}$, the set $\mathfrak{A}(\alpha) = \{f(\alpha) \mid f \in \mathfrak{A}\}$ is an additive subgroup of $\widehat{V}$ but not an ideal of $\widehat{V}$. However, $\widehat{V}$ and V are two valuation domains, with same value group Γ, embedded in the same complete lattice Γ^*, and the lower bound $\inf\{v\big(f(\alpha)\big) \mid f \in \mathfrak{A}\}$ does correspond to the ideal generated in $\widehat{V}$ by $\mathfrak{A}(\alpha)$.
2) For a discrete valuation domain V, each ideal of V is principal and the value function $\Psi_{\mathfrak{A}}$ takes its values in the value group Γ itself. In this case $\Psi_{\mathfrak{A}}$ is upper semi-continuous: for each $\alpha \in \widehat{E}$, there exists a neighborhood U of α such that $\Psi_{\mathfrak{A}}(\beta) \leq \Psi_{\mathfrak{A}}(\alpha)$ for $\beta \in U$ [9, Lemma 1.1]. The following example shows that this property does not generalize, even in the case of a rank-one valuation domain.

Example 3.2. Let V be a valuation domain with value group $\mathbb{R}$ and E be the precompact subset of V formed by a sequence $\{x_n\}$ (converging to 0) such that $v(x_n) = n$. It follows from Corollary 1.11, that for each integer $n \geq 1$, there exists a polynomial $f_n \in \mathrm{Int}(E, V)$ such that

$$v\big(f_n(x)\big) = 0 \text{ if } v(x) < n, \text{ and } v\big(f_n(x)\big) = 1/n \text{ if } v(x) \geq n.$$

Let $\mathfrak{A}$ be the ideal of $\mathrm{Int}(E, V)$ generated by the polynomials f_n. It is not difficult to verify that the ideal of values $\Psi_{\mathfrak{A}}(0)$ is the maximal ideal $\mathfrak{m}$ of V, whereas, for each x such that $v(x) \geq 1$, letting $N = [v(x)]$ (the entire part of $v(x)$), the ideal of values $\Psi_{\mathfrak{A}}(x)$ is the ideal $I_{1/N}$, where $I_{1/N} = \{a \in V \mid v(a) \geq 1/N\}$. However, it is clear that we have $\mathfrak{m} < I_{1/N}$.

Finitely generated ideals. If $\mathfrak{A} = (g_1, \ldots, g_k)$ is a finitely generated ideal, then, for each $\alpha \in E$, the ideal of values $\mathfrak{A}(\alpha)$ is principal, and hence, one can interpret the value function $\Psi_{\mathfrak{A}}$ as a function from $\widehat{E}$ to $\Gamma \cup \{\infty\}$, with

$$\Psi_{\mathfrak{A}}(\alpha) = \inf_{1 \leq i \leq k} \{v\big(g_i(\alpha)\big)\}.$$

Moreover, if $\mathfrak{A}$ is unitary (that is, contains non-zero constants), then $\Psi_{\mathfrak{A}}$ takes its values in Γ (that is, one never has $\Psi_{\mathfrak{A}}(\alpha) = \infty$). As each g_i is continuous, we have the following:

Lemma 3.3. *Let $\mathfrak{A}$ be a finitely generated unitary ideal. Then $\Psi_{\mathfrak{A}}$ is a locally constant function from $\widehat{E}$ to Γ. In particular, if E is precompact, then $\Psi_{\mathfrak{A}}$ takes only finitely many distinct values.*

We shall prove below that, conversely, for a unitary ideal $\mathfrak{A}$, $\Psi_{\mathfrak{A}}$ is locally constant if and only if $\mathfrak{A}$ is finitely generated (in fact, generated by two elements) [Corollary 3.8]. But right now, we can derive the following remark:

Remark 3.4. The value function of a finitely generated ideal is entirely determined by its values on E.

Skolem properties. The Skolem properties measure to what extent an ideal $\mathfrak{A}$ is characterized by its ideals of values $\mathfrak{A}(a)$ for $a \in E$. For a finitely generated ideal, these ideals entirely determine the value function $\Psi_{\mathfrak{A}}$, but in general, this is not the case, thus we more generally ask to what extent an ideal is characterized by its value function (on $\widehat{E}$). In other words, we ask if the equality $\Psi_{\mathfrak{A}} = \Psi_{\mathfrak{B}}$ implies $\mathfrak{A} = \mathfrak{B}$. Let us immediately note that the ideals respectively generated by 1 and by $1 + tX$, where $t \in \mathfrak{m}$, have the same value function, although they are distinct. Hence, as always in the local case, we restrict ourselves to unitary ideals (that is, the ideals containing non-zero constants). Let us recall the definitions [4, 5]:
— One says that $\mathrm{Int}(E, V)$ satisfies the *almost super Skolem property* if the unitary ideals of $\mathrm{Int}(E, V)$ are characterized by their value functions (on $\widehat{E}$).
— One says that $\mathrm{Int}(E, V)$ satisfies the *almost strong Skolem property* if the finitely generated unitary ideals of $\mathrm{Int}(E, V)$ are characterized by their ideals of values (on E, indeed, for the finitely generated ideals, one may restrict the value functions to E.).

— One says that $\mathrm{Int}(E,V)$ satisfies the *almost Skolem property* if, for a finitely generated unitary ideal $\mathfrak{A}$, the condition $\mathfrak{A}(\alpha) = V$, for each $\alpha \in E$, implies $\mathfrak{A} = \mathrm{Int}(E,V)$.

Clearly, each property implies the following one. The corresponding Skolem properties have the same definitions without any restriction to unitary ideals. The last one was originally introduced by Skolem (in the case of the ring $\mathrm{Int}(\mathbb{Z})$ of integer-valued polynomials on $\mathbb{Z}$). In general, the (non-finitely generated) ideals of $\mathrm{Int}(E,V)$ cannot be characterized by their ideals of values on E only (as illustrated by the example of an ideal of the type $\mathfrak{P}_{\mathfrak{p},\alpha}$ for $\alpha \in \widehat{E} \setminus E$).

For a precompact subset E of K, it follows from Theorem 2.2 that the maximal unitary ideals of $\mathrm{Int}(E,V)$ are the ideals $\mathfrak{P}_{\mathfrak{m},\alpha} = \{f \in \mathrm{Int}(E,V) \mid f(\alpha) \in \widehat{\mathfrak{m}}\}$ for $\alpha \in \widehat{E}$. For a finitely generated ideal, the condition $\mathfrak{A}(\alpha) = V$ for each $\alpha \in E$ implies $\Psi_{\mathfrak{A}} = 0$, that is, $\mathfrak{A}$ is not contained in any $\mathfrak{P}_{\mathfrak{m},\alpha}$. In other words, $\mathrm{Int}(E,V)$ satisfies the almost Skolem property (as in the Noetherian case [4, Prop. VII.1.12]). In fact, we even have the following, generalizing [9, Theorem 1.3] and along the line of [5, Proposition 4.2]:

Theorem 3.5. *Let E be a precompact subset of K. Then $\mathrm{Int}(E,V)$ satisfies the almost super Skolem property: if $\mathfrak{A}$ is a unitary ideal of $\mathrm{Int}(E,V)$, and f an integer-valued polynomial, then $f \in \mathfrak{A}$ if and only if $\Psi_f \geq \Psi_{\mathfrak{A}}$.*

Proof. By definition of the function $\Psi_{\mathfrak{A}}$, if $f \in \mathfrak{A}$ then $\Psi_f \geq \Psi_{\mathfrak{A}}$. We must prove the converse. Since $\mathfrak{A}$ is unitary, it contains a non-zero constant a. Suppose that $\Psi_f \geq \Psi_{\mathfrak{A}}$: for each $\alpha \in \widehat{E}$, we have

$$f(\alpha) \geq \inf\{v\big(g(\alpha)\big) \mid g \in \mathfrak{A}\},$$

that is $f(\alpha)$ belongs to the ideal $\mathfrak{A}(\alpha)\widehat{V}$ (generated by the set of values $\mathfrak{A}(\alpha)$ on $\widehat{V}$). As V is dense in $\widehat{V}$, and hence, $\mathfrak{A}(\alpha)$ is dense in $\mathfrak{A}(\alpha)\widehat{V}$, there exists a polynomial $g_\alpha \in \mathfrak{A}$, such that $v\big(f(\alpha) - g_\alpha(\alpha)\big) \geq v(a)$. As f and g_α are continuous, there is a clopen neighborhood U_α of α such that $v\big(f(z) - g_\alpha(z)\big) \geq v(a)$, for $z \in U_\alpha$. Since $\widehat{E}$ is compact, it can be covered by finitely many clopen sets, say $U_1, \ldots, U_s$, corresponding to polynomials $g_1, \ldots, g_s$ in $\mathfrak{A}$ such that $v\big(f(z) - g_i(z)\big) \geq v(a)$ for $z \in U_i$. We can moreover require these clopen sets to have pairwise empty intersections. From the Stone-Weierstrass theorem [Theorem 1.10], the characteristic function Ψ_i of each clopen set U_i can be uniformly approximated by an integer-valued polynomial φ_i, in such a way that $v\big(\Psi_i(z) - \varphi_i(z)\big) \geq v(a)$ for all $z \in \widehat{E}$. We then have $v\big(f(z) - \sum_i g_i(z)\varphi_i(z)\big) \geq v(a)$ for all $z \in \widehat{E}$. In other words, letting $h = f - \sum_i g_i\varphi_i$, the polynomial $a^{-1}h$ is integer-valued. As $a \in \mathfrak{A}$, it follows that $h = a(a^{-1}h)$ belongs to $\mathfrak{A}$. Finally, writing $f = \sum_i g_i\varphi_i + h$, we conclude that we have $f \in \mathfrak{A}$. □

A fortiori, we thus have also the following, generalizing the one-dimensional case [5, Theorem 4.7]:

Corollary 3.6. *Let E be a precompact subset of K. Then* $\mathrm{Int}(E,V)$ *satisfies the almost strong Skolem property.*

The two generators property. From the almost strong Skolem property, we now derive that $\mathrm{Int}(E,V)$ satisfies the two generators property. If $\mathfrak{B}$ is a finitely generated ideal, it is easy to see that $\mathfrak{B}=\varphi\mathfrak{A}$ where $\mathfrak{A}$ is a unitary ideal and φ is a polynomial (as in the case of the ring $\mathrm{Int}(D)$ of integer-valued polynomials on a domain D [4, Lemma VI.1.2]). One can thus focus on unitary ideals.

Proposition 3.7. *Let E be a precompact subset of K and let Ψ be a locally constant function from $\widehat{E}$ to Γ^*. Then Ψ is the value function of one and only one finitely generated unitary ideal $\mathfrak{A}$. Moreover $\mathfrak{A}$ is two-generated; more precisely, there exists a polynomial $f\in\mathrm{Int}(E,V)$ and a non-zero constant $a\in V$ such that $\Psi=\Psi_f=\Psi_{\mathfrak{A}}$, where $\mathfrak{A}=(a,f)$.*

Proof. From the Stone-Weierstrass theorem [Corollary 1.11] it follows that we can find a polynomial $f\in\mathrm{Int}(E,V)$ such that $v\big(f(\alpha)\big)=\Psi(\alpha)$ for each $\alpha\in E$. By continuity, one thus have $\Psi_f=\Psi$. Choosing a non-zero $a\in V$ such that $v(a)\geq\max_{\alpha\in E}\{v\big(f(\alpha)\big)\}$, and letting $\mathfrak{A}=(a,f)$, then $\mathfrak{A}$ is unitary and $\Psi_{\mathfrak{A}}=\Psi_f=\Psi$. Finally, $\mathfrak{A}$ is unique since $\mathrm{Int}(E,V)$ satisfies the almost super Skolem property, hence a fortiori, the almost strong Skolem property [Theorem 3.5]. □

From the almost super Skolem property, we also derive immediately the following, as announced above:

Corollary 3.8. *Let E be a precompact subset of K. The value function $\Psi_{\mathfrak{A}}$ of a unitary ideal $\mathfrak{A}$ is locally constant if and only if $\mathfrak{A}$ is finitely generated.*

In particular, let us consider the value function Ψ of the prime ideal $\mathfrak{P}_{\mathfrak{p},\alpha}=\{f\in\mathrm{Int}(E,V)\mid f(\alpha)\in\widehat{\mathfrak{p}}\}$. It is such that $\Psi(\alpha)>0$ and $\Psi(\beta)=0$, for $\beta\neq\alpha$. Hence Ψ is not locally constant unless α is isolated in $\widehat{E}$. We thus derive the following:

Corollary 3.9. *Let E be a precompact subset of K. For each non-zero prime ideal $\mathfrak{p}$ of V and each $\alpha\in\widehat{E}$, the prime ideal $\mathfrak{P}_{\mathfrak{p},\alpha}$ is finitely generated if and only if α is isolated in $\widehat{E}$.*

In fact, we have the strong two-generators property: every finitely generated ideal of $\mathrm{Int}(E,V)$ admits a system of two generators and one of them can be arbitrarily chosen.

Theorem 3.10. *Let E be a precompact subset of K and $\mathfrak{A}$ be a finitely generated ideal of* $\mathrm{Int}(E,V)$*. Then, for each non-zero $g\in\mathfrak{A}$, there exists a polynomial $h\in\mathfrak{A}$ such that $\mathfrak{A}=(g,h)$.*

Proof. As above, we may restrict ourselves to the case of a unitary ideal. From Proposition 3.7, there exists $f\in\mathrm{Int}(E,V)$ such that $\Psi_{\mathfrak{A}}=\Psi_f$. From Theorem 3.5 it follows that, in fact, $f\in\mathfrak{A}$. Given g, there is at most one constant $a \in K$, such that g and $f+a$ are not coprime in $K[X]$. Chose a such that $v(a) \geq \max_{\alpha \in E}\{v\big(f(\alpha)\big)\}$ and such that g and $f+a$ are coprime. Let $h = f+a$ and $\mathfrak{B} = (g,h)$. Then $\mathfrak{B}$ is a unitary ideal and $\Psi_{\mathfrak{A}} = \Psi_{\mathfrak{B}}$. Thus $\mathfrak{A} = \mathfrak{B}$. □

4. The Prüfer property

It is well-known that the classical ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z})$ is a Prüfer domain. In general, for a (fractional) subset E of a domain D, we clearly have the following necessary conditions:
If $\mathrm{Int}(E,D)$ is a Prüfer domain, then:
— D itself (which is an homomorphic image of $\mathrm{Int}(E,D)$) is a Prüfer domain,
— for each prime ideal $\mathfrak{p}$ of D, $\mathrm{Int}(E,D_{\mathfrak{p}})$ (which is an overring of $\mathrm{Int}(E,D)$) is a Prüfer domain.
This allows us to focus on the case where $D = V$ is a valuation domain.

It is known that, if V is a discrete valuation domain with finite residue field, then $\mathrm{Int}(V)$ is a Prüfer domain. We may generalize this result, for a precompact subset of any valuation domain, using both the description of the maximal ideals of $\mathrm{Int}(E,V)$ (section 2) and the Stone-Weierstrass theorem (section 1); the proof is thus almost the same as in the classical case [4, Lemma VI.1.4]:

Theorem 4.1. *Let E be a precompact subset of K. Then* $\mathrm{Int}(E,V)$ *is a Prüfer domain.*

Proof. Let $\mathfrak{A} = (f_1,\dots,f_r)$ be a finitely generated ideal of $\mathrm{Int}(E,V)$. We have to show that $\mathfrak{A}$ is invertible. Since there is a polynomial φ such that $\mathfrak{A} = \varphi\mathfrak{B}$ and $\mathfrak{B}$ is a unitary ideal, we may assume that $\mathfrak{A}$ is itself unitary. Consider $\mathfrak{A}^{-1} = \{f \in K[X] \mid f\mathfrak{A} \subseteq \mathrm{Int}(E,V)\}$. We wish to prove that $\mathfrak{A}\mathfrak{A}^{-1} = \mathrm{Int}(E,V)$. Since $\mathfrak{A} \subseteq \mathfrak{A}\mathfrak{A}^{-1} \subseteq \mathrm{Int}(E,V)$, it suffices to show that, for each $\alpha \in \widehat{E}$, $\mathfrak{A}\mathfrak{A}^{-1}$ is not contained in $\mathfrak{P}_{\mathfrak{m},\alpha}$ (see Corollary 2.4).

Let $\alpha \in \widehat{E}$ and let $\gamma = \inf_{1\leq i\leq r} v\left(f_i(\alpha)\right) \in \Gamma$. By continuity, there is a clopen neighborhood U of α in $\widehat{E}$ such that $v\left(f(x)\right) \geq \gamma$ for each $f \in \mathfrak{A}$ and each $x \in U$. From the Stone-Weierstrass theorem [Corollary 1.11], there exists a polynomial $h \in K[X]$ such that $v(h(x)) = -\gamma$ for $x \in U$ and $v(h(x)) \geq 0$ for $x \in \widehat{E} \setminus U$. Obviously, $h \in \mathfrak{A}^{-1}$. Now, let $f_0 \in \mathfrak{A}$ be such that $v(f_0(\alpha)) = \gamma$. Then $v\left(h(\alpha)f_0(\alpha)\right) = 0$. Consequently, $hf_0 \in \mathfrak{A}\mathfrak{A}^{-1}$ while $hf_0 \notin \mathfrak{P}_{\mathfrak{m},\alpha}$. □

Remark 4.2. In fact, if V has finite dimension, Theorem 4.1 follows also directly from the two-generators property, since an n-dimensional integrally closed domain such that every finitely generated ideal requires at most $n+1$ generators is a Prüfer domain [11].

From Theorem 4.1 the precompactness of E is sufficient for $\mathrm{Int}(E,V)$ to be a Prüfer domain. One question remains: to what extent is it necessary for this property and also for all the previous results? We have some partial answers:

— If $\mathrm{Int}(E,V)$ is a Prüfer domain, then $\mathrm{Int}(E,V)$ has the almost strong Skolem property. Indeed, each finitely generated ideal of $\mathrm{Int}(E,V)$ is invertible, a fortiori is divisorial, and one knows that each unitary divisorial ideal $\mathfrak{A}$ is*Skolem closed* (that is, for each $f \in \mathrm{Int}(E,V)$, $\Psi_f \geq \Psi_{\mathfrak{A}}$ implies $f \in \mathfrak{A}$) [5, Lemma 3.7] .
— If V is a rank-one valuation domain, then $\mathrm{Int}(E,V)$ is dense in $\mathcal{C}(\widehat{E},\widehat{V})$ if and only if E is precompact [6, Theorem 3.3].
— If V is a discrete valuation domain, then $\mathrm{Int}(E,V)$ has the almost strong Skolem property if and only if E is precompact [5, Theorem 4.7].
From Theorem 4.1, Remark 4.2 and the facts above, it follows that we have:

Corollary 4.3. *Let V be a discrete valuation domain. The following assertions are equivalent:*
(i) *E is a precompact subset of K.*
(ii) $\mathrm{Int}(E,V)$ *is dense in* $\mathcal{C}(\widehat{E},\widehat{V})$
(iii) $\mathrm{Int}(E,V)$ *has the almost strong Skolem property.*
(iv) $\mathrm{Int}(E,V)$ *satisfies the two-generators property.*
(v) $\mathrm{Int}(E,V)$ *is a Prüfer domain.*

Finally, we close this paper with an example of a very high dimensional valuation domain for which there are no infinite precompact subset, and where it precisely turns out that only the finite subsets are suitable for the Prüfer property.

Example 4.4. Let $\mathcal{E}$ be the ring of entire functions and let E be a subset of $\mathcal{E}$. Then $\mathrm{Int}(E,\mathcal{E})$ is a Prüfer domain if and only if E is finite.

Proof. Recall that $\mathcal{E}$ is an infinite-dimensional Bézout domain (that is, such that each finitely generated ideal is principal) and that its quotient field is the field $\mathcal{M}$ of meromorphic functions (see [12, Chapter VIII], or [13]).

For $f \in \mathcal{E}$, we let $Z(f) = \{x \in \mathbb{C} \mid f(x) = 0\}$ be the set of zeros of f; it is known that $Z(f)$ is a closed and discrete subset of $\mathbb{C}$. One can then show that the maximal ideals of $\mathcal{E}$ are of the form

$$\mathfrak{m}_{\mathcal{U}} = \{f \in \mathcal{E} \mid Z(f) \in \mathcal{U}\}$$

where $\mathcal{U}$ is an ultrafilter of $\mathbb{C}$ which contains such a closed and discrete subset. In the case where $\mathcal{U}$ is principal, that is, is formed by the subsets containing an element $x \in \mathbb{C}$, then $\mathfrak{m}_{\mathcal{U}}$ is simply the set of functions such that $f(x) = 0$, and it is an height-one principal ideal. On the contrary, if $\mathcal{U}$ is a non-principal ultrafilter, the height of $\mathfrak{m}_{\mathcal{U}}$ is infinite (see [3, Chapter V, §1, ex. 12]).

As $\mathcal{E}$ is a Bézout domain, and hence a Prüfer domain, it follows that, for each maximal ideal $\mathfrak{m}$ of $\mathcal{E}$, the localization $\mathcal{E}_{\mathfrak{m}}$ is a valuation domain. In particular, for a non-principal ultrafilter we thus obtain an infinite dimensional valuation domain:

$$V_{\mathcal{U}} = \mathcal{E}_{\mathfrak{m}_{\mathcal{U}}} = \left\{ \frac{f}{g} \in \mathcal{M} \mid f,\, g \in \mathcal{E},\, Z(g) \notin \mathcal{U} \right\}.$$

As $\mathcal{E}$ is a Prüfer domain, it follows also that, for every finite subset E of $\mathcal{M}$, $\mathrm{Int}(E, \mathcal{E})$ is a Prüfer domain [15]. On the other hand, let E be an infinite subset of $\mathcal{E}$. We claim that, for each non-principal ultrafilter $\mathcal{U}$, $\mathrm{Int}(E, V_{\mathcal{U}})$ is not a Prüfer domain (in particular, it follows that E is not precompact; this is also a consequence of the fact that $V_{\mathcal{U}} \setminus \{0\}$ has no cofinal subset [13, Lemma 4]). Consequently, $\mathrm{Int}(E, \mathcal{E})$, which is contained in $\mathrm{Int}(E, V_{\mathcal{U}})$, cannot be Prüfer.

We first consider the case where $\mathcal{U}$ contains the set of positive integers $\mathbb{N}$. Let $\{\phi_1, \phi_2, \ldots, \phi_n, \ldots\}$ be a countable subset of E. For each $n \in \mathbb{N}$, set $w_n = 1 + \max_{0 \leq i < j \leq n} nv_n(\phi_j - \phi_i)$ (where $v_n(\phi)$ denotes the order of ϕ at n). There exists $f \in \mathcal{E}$ such that, for each $n \in \mathbb{N}$, $v_n(f) = w_n$. Let $\mathfrak{p}$ be the radical in $V_{\mathcal{U}}$ of the principal ideal (f). If g belongs to $\mathfrak{p}$, there is $k \in \mathbb{N}$ such that $g^k \in (f)$, that is, $g^k = f\frac{\varphi}{\psi}$, where $Z(\psi) \notin \mathcal{U}$. As $\mathcal{U}$ is an ultrafilter, the complement of $Z(\psi)$ belongs to $\mathcal{U}$ and so does its intersection with $\mathbb{N}$. As $\mathcal{U}$ is not principal, it does not contain any finite subset, hence, for infinitely many integers, and in particular, for arbitrarily large ones, we have $\psi(n) \neq 0$. For such an n, we have $v_n(f) \leq kv_n(g)$, and if $n \geq k$, we thus have $v_n(f) \leq nv_n(g)$. Consequently, for $i \neq j$, $\phi_j - \phi_i \notin \mathfrak{p}$ since, by construction, for $n \geq i, j$, one has $nv_n(\phi_j - \phi_i) < v_n(f)$. Thus E meets infinitely many distinct classes of $V_{\mathcal{U}}$ modulo $\mathfrak{p}$, and hence, we have the containment $\mathrm{Int}(E, V_{\mathcal{U}}) \subseteq (V_{\mathcal{U}})_{\mathfrak{p}}[X]$ [4, IV.1.20]. Therefore, we can conclude that $\mathrm{Int}(E, V_{\mathcal{U}})$ is not Prüfer.

This proof may be extended to any non-principal ultrafilter $\mathcal{U}$ which contains a closed and discrete subset N of $\mathbb{C}$. Indeed, such an N is a countable subset and we may replace the sequence $\{n\}_{n\in\mathbb{N}}$ by a sequence $\{x_n\}_{n\in\mathbb{N}}$ formed by all the elements of N. □

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