FINITE GENERATION PROPERTIES FOR VARIOUS RINGS OF INTEGER-VALUED POLYNOMIALS

MANJUL BHARGAVA, PAUL-JEAN CAHEN, AND JULIE YERAMIAN

Abstract. For any subset E of a Dedekind domain D, we show the ring $\mathrm{Int}^{\{r\}}(E, D)$ of polynomials that are integer-valued on E together with all their divided differences of order up to r not to be a finitely generated D-algebra, contrary to the ring $\mathrm{Int}_x(E, D)$ of integer-valued polynomials on E having a given nonzero modulus x (which is hence Noetherian, since the domain D is so). Localization properties allow us to focus on valuation domains; furthermore, the consideration of precompact subsets allows us to consider valuation domains V of arbitrary dimension.

Introduction

We consider a domain D with quotient field K and a subset E of K. We suppose that D is not a field, that is $D \neq K$ (in particular, D is infinite). We denote by $\mathrm{Int}(E, D)$ the ring of integer-valued polynomials on E, that is,

$$\mathrm{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}.$$

For short, we write $\mathrm{Int}(D)$ for $\mathrm{Int}(D, D)$. In some parts, we focus on the case where D is a valuation domain (we then denote it by V, thus writing $\mathrm{Int}(E, V)$ for $\mathrm{Int}(E, D)$).

Considering various natural classes of subrings of $\mathrm{Int}(E, D)$, we address the question of their Noetherian properties. Integer-valued polynomials were first considered over $\mathbb{Z}$ and rings of integers of number fields (by Polya and Ostrowski) and the classical ring $\mathrm{Int}(\mathbb{Z})$ is well known to be non-Noetherian. More generally, $\mathrm{Int}(E, D)$ is almost never Noetherian [7, Exercise VI.13]. In particular, $\mathrm{Int}(D)$ is not Noetherian unless it is contained in the ring of polynomials $D'[X]$ with coefficients in the integral closure D' of D [7, Prop VI 2.4.]); thus, if D is integrally closed, $\mathrm{Int}(D)$ is not Noetherian unless it is trivial, that is, $\mathrm{Int}(D) = D[X]$.

Clearly, a necessary condition for a subring R of $\mathrm{Int}(E, D)$, containing D, to be Noetherian is that D itself be Noetherian. However, we are naturally led to consider integer-valued polynomials over non-Noetherian base rings. Indeed, in the classical theory (say over the ring of integers of a number field), localization properties allow one to focus on discrete valuation domains; the fact that such rings are compact is a key factor, but here, the consideration of (precompact) subsets allows one to consider non-Noetherian valuation domains [9] and even valuation domains of arbitrary dimension [10]. It thus seems more pertinent to ask whether the ring R we consider is a finitely generated D-algebra (which is clearly sufficient, although not necessary, for R to be Noetherian when D is so).

In this article, we consider the following two classes of subrings of $\mathrm{Int}(E, D)$:

Key words and phrases. Integer-valued polynomials, finite differences, valuation domain, precompact subset, Noetherian property.

–The rings $\text{Int}^{\{r\}}(E,D)$ of polynomials which are integer-valued on E together with all their *divided differences of order up to* r.

–The rings $\text{Int}_x(E,D)$ of integer-valued polynomials on E *having a given modulus* x.

Both of these classes of rings arose from studies in nonarchimedean analysis [5]. The first class $\text{Int}^{\{r\}}(E,D)$ arose in the context of D-valued continuously differentiable functions on compact subsets E of a complete discrete valuation domain D, while the second class $\text{Int}_x(E,D)$ arose analogously in the study of locally analytic functions on such subsets. The connections with nonarchimedean functions allow us to deduce a number of properties of these two types of rings, and conversely the understanding of properties of these rings may well shed light on the corresponding problems in nonarchimedean analysis. Indeed, this was one of our first motivations in investigating $\text{Int}^{\{r\}}(E,D)$ and $\text{Int}_x(E,D)$, although we believe these rings are also very interesting in their own right, having very natural and elementary definitions.

The first type of ring $\text{Int}^{\{r\}}(E,D)$ is defined to be the ring of all polynomials in $K[x]$ whose *k-th divided differences* are integer-valued (that is, D-valued) on E for all $k \in \{0,\ldots,r\}$. Recall that the ($1st$) *divided difference* Φf of a polynomial f is defined to be

$$\Phi f(X_0,X_1) = \frac{f(X_0)-f(X_1)}{X_0-X_1},$$

and the *k-th divided difference* $\Phi^k f(X_0,\ldots,X_k)$ for $k>1$ is then defined inductively by

$$(1)\qquad \Phi^k f(X_0,\ldots,X_k) = \frac{\Phi^{k-1}f(X_0,\ldots,X_{k-1}) - \Phi^{k-1}f(X_0,\ldots,X_{k-2},X_k)}{X_{k-1}-X_k}.$$

By convention, $\Phi^0 f(X) = f(X)$, and in general one sees that $\Phi^k f(X_0,\ldots,X_k)$ is a symmetric polynomial in the $k+1$ indeterminates $X_0,\ldots,X_k$. In this notation, we thus have

$$(2)\qquad \text{Int}^{\{r\}}(E,D) = \{f \in K[X] \mid \forall k \le r, \Phi^k f(E^{k+1}) \subseteq D\}.$$

One shows that the sets $\text{Int}^{\{r\}}(E,D)$ of polynomials are in fact subrings of $\text{Int}(E,D)$ for all r. Note that in the case $r=0$ we recover the ring $\text{Int}(E,D)$.

The rings $\text{Int}_x(E,D)$ of integer-valued polynomials of modulus x were studied by the third author in her Ph. D. thesis [16], under the name of *Bhargava rings* (as they stem from a talk given by the first author at the *Deuxième rencontre internationale sur les polynômes à valeurs entières*, Marseille Luminy, June 2000). These rings are defined as follows. For a subset E of the quotient field K of a domain D, let

$$(3)\qquad \text{Int}_x(E,D) = \{f \in K[X] \mid \forall a \in E, f(xX+a) \in D[X]\}.$$

Just as with $\text{Int}^{\{r\}}(E,D)$, the set $\text{Int}_x(E,D)$ forms a subring of $\text{Int}(E,D)$ for any $x \in D$. Note that the case $x=0$ again recovers $\text{Int}(E,D)$.

Thus the rings $\text{Int}^{\{r\}}(E,D)$ and $\text{Int}_x(E,D)$ naturally extend (and in very different ways, as we will see) the well-studied ring $\text{Int}(E,D)$. In a first section, we state and prove some generalities on both these classes of rings, and in particular, we establish the localization properties which allow us to focus on valuation domains.

Next, for a valuation domain V with no restrictive hypothesis other than the existence of an infinite precompact subset E, we give constructions of regular bases for both rings $\text{Int}^{\{r\}}(E,V)$ (of polynomials having integer-valued divided differences on E of order up to r) and $\mathrm{Int}_x(E,V)$ (of integer-valued polynomials of modulus x). These two constructions require, respectively, two appropriately adapted versions of the notion of *v-ordering*, first introduced in [3] by the first author, and studied and generalized to other contexts in [5], [8, 9], [10], and [16]. The two types of v-orderings we require are called "r-removed v-orderings" and "v-orderings of order h" respectively, and the associated constructions of regular bases lead to explicit descriptions of the additive structures of these rings.

We then address the Noetherian and finite generation properties. In the case $x \neq 0$ and V a discrete valuation domain, we show that the ring $\mathrm{Int}_x(E,V)$ of integer-valued polynomials on E having modulus x is a finitely generated V-algebra, and is hence Noetherian. By globalization, we therefore obtain:

Theorem 0.1. *Suppose D is a Dedekind domain, $E \subset D$ a subset, and $x \neq 0$ an element of D. Then the ring $\mathrm{Int}_x(E,D)$ is finitely generated over D and is hence Noetherian.*

Our methods are effective and allow us to construct an explicit set of D-algebra generators for $\mathrm{Int}_x(E,D)$. The arguments also allow us to replace D more generally by a Krull domain, and E by any locally precompact subset of the fraction field of D. The central ingredient in obtaining Theorem 0.1 is a proof (in the local case where $D = V$ is a discrete valuation domain) of the existence of a *periodic* v-ordering of order α of E for any precompact subset $E \subset V$.

If V is not a discrete valuation domain, however, then we show that a precompact subset $E \subset V$ does not necessarily possess a periodic v-ordering of order α, and indeed $\mathrm{Int}_x(E,D)$ need not be Noetherian or finitely generated in such a case.

In stark contrast to $\mathrm{Int}_x(E,D)$, we then prove that the ring $\mathrm{Int}^{\{r\}}(E,D)$, consisting of those polynomials whose k-th divided differences for $k \leq r$ are integer-valued on E, is not a finitely generated D-algebra nor is it Noetherian. We recall that the classical argument to show that $\mathrm{Int}(D)$ is not Noetherian uses a property of separation of points [7, §III.4]): for $a \neq b$ in D, and any maximal ideal $\mathfrak{m}$, there exists a polynomial $f \in \mathrm{Int}(D)$ such that

$$\begin{cases} f(a) \equiv 0 \pmod{\mathfrak{m}} \\ f(b) \equiv 1 \pmod{\mathfrak{m}}\ . \end{cases}$$

The situation is similar for polynomials with integer-valued derivatives of order up to r (if f is integer-valued, some power of f is such that its derivative is also integer-valued). However, the same argument does not apply directly to $\mathrm{Int}^{\{r\}}(E,V)$: if the divided difference $\frac{f(Y)-f(X)}{Y-X}$ is integer-valued, then $a \equiv b \pmod{\mathfrak{m}}$ clearly implies $f(a) \equiv f(b) \pmod{\mathfrak{m}}$. Nevertheless we show that, for any precompact subset E of any valuation domain V, the ring $\mathrm{Int}^{\{r\}}(E,V)$ is neither Noetherian nor a finitely generated V-algebra, by using a separation argument involving an r-th divided difference. In particular, this holds for every subset of a discrete valuation domain with finite residue field. By globalization, we then obtain:

Theorem 0.2. *Suppose D is a Dedekind domain with finite residue fields and $E \subset D$ any subset. Let $r \geq 0$ be any integer. Then the ring $\mathrm{Int}^{\{r\}}(E,D)$ is not finitely generated over D and is not Noetherian.*

It follows, for example, that $\mathrm{Int}^{\{r\}}(E,\mathbb{Z})$ is not Noetherian. More generally, our arguments show that the same applies to a Krull domain D and any locally precompact subset E of D (that is, any subset E that is precompact in $D_{\mathfrak{p}}$ for every height one prime ideal $\mathfrak{p}$ of D). The result could also be extended to the case of a locally precompact subset E of a Prüfer domain D, whenever integer-valued polynomials have good localization behavior (that is, $\text{Int}(E, D)_{\mathfrak{m}} = \text{Int}(E, D_{\mathfrak{m}})$ for each maximal ideal $\mathfrak{m}$ of D).

We show however that, for an infinite subset E, the ring $\text{Int}^{\{r\}}(E, D)$ satisfies the ascending chain condition on principal ideals (for short ACCP) if and only if this is the case for the base ring D (as for integer-valued polynomials [7, Proposition VI.2.9]) whereas, if E is finite, $\text{Int}^{\{r\}}(E, D)$ never satisfies ACCP.

Finally, we consider a simultaneous generalization of the two rings $\text{Int}^{\{r\}}(E, D)$ and $\text{Int}_x(E, D)$, namely

$$\text{Int}_x^{\{r\}}(E, D) = \{f \in K[X] \mid \forall k \leq r, \forall a \in E^{k+1}, \Phi^k f(xE^{k+1} + a) \subseteq D\}. \tag{4}$$

The techniques developed to deal with the two classes of rings individually may in fact be combined to prove analogous finite generation theorems for the rings $\text{Int}_x^{\{r\}}(E, D)$. These results are discussed in the final section.

1. Definitions and generalities

1.1. **Divided differences.** Integer-valued finite differences were studied first over $\mathbb{Z}$ by Carlitz [11] and then over more general domains by Barsky [2] and others [14, 15]. Recall that the first finite difference of a polynomial f is defined to be

$$\Delta_h f = \frac{f(X+h) - f(X)}{h},$$

the successive finite differences then being defined inductively by

$$\Delta_{h_1, h_2 \ldots, h_r} f = \frac{\Delta_{h_1, h_2 \ldots, h_{r-1}} f(X + h_r) - \Delta_{h_1, h_2 \ldots, h_{r-1}} f(X)}{h_r}$$

Clearly, finite differences are not appropriate for the consideration of subsets: the condition that the first finite difference is integer-valued means that for $a, h \in D$, $\frac{f(a+h)-f(a)}{h} \in D$; for a subset E, it may happen that $a, h \in E$, and $(a + h) \notin E$.

Following the notion of *interpolar functions* attributed by Cauchy to Ampere [12], we consider the *k-th divided differences* $\Phi^k f$ of a polynomial f with coefficients in a ring A (not necessarily a domain), defined by $\Phi^0 f(X_0) = f(X_0)$ and then inductively by Equation (1). In particular, we see that $\Delta_h f = \Phi f(X, X + h)$.

The k-th divided differences of a polynomial f have many spectacular properties, and they have played an important role in a number of subjects in mathematics. We list here just a few of these properties that we will need, most of them due to Cauchy [12].

1.1.1. If f is a polynomial of degree n, then $\Phi^k f(X_0, ..., X_k)$ is a symmetric polynomial of degree $n - k$ in $k + 1$ indeterminates [12, Theorem III], given explicitly by

$$\Phi^k f(X_0, \ldots, X_k) = \sum_{0 \leq i \leq k} \frac{f(X_i)}{\prod_{j \neq i}(X_i - X_j)}.$$

In particular, $\Phi^n f$ is a constant, and for $k > n$, $\Phi^k f = 0$.

1.1.2. The divided difference operators Φ^k are linear [12, Theorem V]: if $f = \alpha\varphi + \beta\chi$ with $\alpha, \beta \in A$, then

$$\Phi^k f = \alpha\,\Phi^k\varphi + \beta\,\Phi^k\chi.$$

1.1.3. For $f, g \in A[X]$, we have the product formula:

$$\Phi^k(fg)(X_0, \dots, X_k) = \sum_{i=0}^{k} \Phi^i f(X_0, \dots, X_i)\Phi^{k-i} g(X_i, \dots, X_k).$$

1.1.4. Suppose $g_1, \dots, g_n$ are each polynomials in 1 variable and f is a polynomial in n variables. Let $h(X) = f(g_1(X), \dots, g_n(X))$, and let $\Phi_i f$ denote the divided difference of f with respect to its i-th argument. Then we have the "chain rule"

$$\Phi h(X,Y) = \sum_{i=1}^{n} \Phi_i f(g_1(X), \cdots, (g_i(X), g_i(Y)), \dots, g_n(Y)) \cdot \Phi g_i(X,Y).$$

1.1.5. For a polynomial f, and r values $a, b, c, \dots, h \in A$ [12, Formula (3)]:

$$\begin{aligned} f(X) &= f(a) + (X-a)\Phi^1 f(a,b) + (X-a)(X-b)\Phi^2 f(a,b,c) + \dots \\ &\quad \dots + (X-a)(X-b)\dots(X-h)\Phi^r f(a,b,c,\dots,h,X). \end{aligned}$$

1.1.6. In particular, for a degree n polynomial, with $a = b = c = \dots = h$, we obtain an analogue of Taylor's formula:

$$f(X) = f(a) + (X-a)\Phi^1 f(a,a) + \dots + (X-a)^n \Phi^n f(a,a,\dots,a).$$

(Note that $\Phi^n f$ is a constant).

1.1.7. Comparing with the classical formula, and denoting the k-th derivative of f by $f^{(k)}$, we conclude with [12, Theorem IV]:

$$f^{(k)}(X) = k!\Phi^k f(X, \dots, X).$$

1.2. **Rings of integer-valued divided differences.** We now come back to a domain D, with quotient field K. It follows from the product formula 1.1.3 that the polynomials (with coefficients in K) that are integer-valued on E, together with all their k-th divided differences for $k \le r$, form a ring. We denote this ring by $\mathrm{Int}^{\{r\}}(E,D)$ as in (2).

Writing $\mathrm{Int}^{\{0\}}(E,D) = \mathrm{Int}(E,D)$, we can also define $\mathrm{Int}^{\{r\}}(E,D)$ inductively:

$$\mathrm{Int}^{\{r\}}(E,D) = \{f \in \mathrm{Int}(E,D) \mid \forall a \in E, \Phi f(a,X) \in \mathrm{Int}^{\{r-1\}}(E,D)\}. \tag{5}$$

Indeed, viewing $\Phi f(a,X)$ as a polynomial in one indeterminate, one may write

$$\Phi^2 f(a, X_1, X_2) = \frac{\Phi f(a, X_2) - \Phi f(a, X_1)}{X_2 - X_1} = \Phi(\Phi f(a, X_1)),$$

and more generally,

$$\Phi^k f(a, X_1, \dots, X_k) = \Phi^{k-1}(\Phi f(a, X_1)).$$

Comparison with finite differences and derivatives. Integer-valued derivatives or finite differences have been studied in several articles [6, 11, 13, 14, 15] (for a survey see [7, chapter IX]). One denotes by $\mathrm{Int}^{(r)}(D)$ (resp. $\mathrm{Int}^{[r]}(D)$) the ring of polynomials that are integer-valued on the domain D together with their derivatives (resp. finite differences) of order up to r. Also one writes $\mathrm{Int}^{(\infty)}(D)$ (resp. $\mathrm{Int}^{[\infty]}(D)$) for the ring of polynomials which are integer-valued together with their derivatives (resp. finite differences) of all order. It is known that one has the (most often strict) containment $\mathrm{Int}^{[r]}(D) \subseteq \mathrm{Int}^{(r)}(D)$ [7, Proposition IX.1.3]. Now, what about $\mathrm{Int}^{\{r\}}(D)$?

For $r = 1, \mathrm{Int}^{\{1\}}(D) = \mathrm{Int}^{[1]}(D)$. Indeed, $\Phi f(a,b) = \frac{f(b)-f(a)}{b-a}$ can be viewed as the finite difference $\Delta_h f(a) = \frac{f(a+h)-f(a)}{h}$ with $h = b-a$. However, $\mathrm{Int}^{\{r\}}(D)$ and $\mathrm{Int}^{[r]}(D)$ differ in general. Let us give an example.

Example 1.1. Let $f = \frac{X(X-1)(X-2)(X-3)}{2}$. Then $f \in \mathrm{Int}^{[\infty]}(\mathbb{Z})$ but $f \notin \mathrm{Int}^{\{2\}}(\mathbb{Z})$. Indeed, the derivative f' is such that $f' \in \mathbb{Z}[X]$, thus the successive derivatives have their coefficients in $\mathbb{Z}$. Hence $f \in \mathrm{Int}^{(\infty)}(\mathbb{Z})$, while it is known that $\mathrm{Int}^{[\infty]}(\mathbb{Z}) = \mathrm{Int}^{(\infty)}(\mathbb{Z})$ [11, Theorem 1]. However, it follows from 1.1.1 that

$$\Phi^{(2)} f(0,2,4) = \frac{f(0)}{2\times 4} - \frac{f(2)}{2\times 2} + \frac{f(4)}{4\times 2} = \frac{3}{2} \notin \mathbb{Z}.$$

Therefore, for $k \geq 2$, $f \in \mathrm{Int}^{[k]}(\mathbb{Z})$ but $f \notin \mathrm{Int}^{\{k\}}(\mathbb{Z})$.

Nevertheless, there is a containment:

Proposition 1.2. *For each r, $\mathrm{Int}^{\{r\}}(D) \subseteq \mathrm{Int}^{[r]}(D)$.*

Proof. We can express the n-th finite difference $\Delta_{h_1,\ldots,h_n} f(X)$ in terms of various n-th divided differences of f; namely, we have the formula

(6)
$$\Delta_{h_1,\ldots,h_n} f(X) = \sum_{\sigma \in S_n} \Phi^n f(X, X+h_{\sigma(1)}, X+h_{\sigma(1)}+h_{\sigma(2)}, \ldots, X+h_{\sigma(1)}+\cdots+h_{\sigma(n)}),$$

where the sum is over all permutations σ in the symmetric group S_n on $\{1, 2, \ldots, n\}$. For $n = 1$ we have $\Delta_{h_1} f(X) = \Phi f(X, X+h_1)$, and the identity (6) then follows by induction using the chain rule 1.1.4 and the fact that the n-th divided difference of f is symmetric in its $n+1$ arguments.

We conclude that if all n-th divided differences of f are integer-valued, then so are all n-th finite differences. □

This containment may be strict for $r \geq 2$. Indeed, it follows from Example 1.1 that it is strict for all $r \geq 2$ when $D = \mathbb{Z}$. Another argument is that $\mathrm{Int}^{\{\infty\}}(D)$ (the ring of polynomials that are integer-valued together with their divided differences of all orders) is trivial, that is, $\mathrm{Int}^{\{\infty\}}(D) = D[X]$ (this follows from Taylor's formula [1.1.6]), whereas usually $\mathrm{Int}^{[\infty]}(D)$ is not ([7, lemma IX.2.10]).

Remark 1.3. Over a domain D, we derive the containments

$$\mathrm{Int}^{\{r\}}(D) \subseteq \mathrm{Int}^{[r]}(D) \subseteq \mathrm{Int}^{(r)}(D).$$

Now derivatives (contrarily to finite differences) make sense on subsets. Denoting by $\mathrm{Int}^{(r)}(E, D)$ the ring of polynomials that are integer-valued on a subset E together with their first r derivatives, it follows immediately from 1.1.7 that we more generally have the containment

$$\mathrm{Int}^{\{r\}}(E,D) \subseteq \mathrm{Int}^{(r)}(E,D).$$

Again, this containment is most often strict, even for $r = 1$. One could argue, as in [7, IX.2], that polynomials in $\mathrm{Int}^{(r)}(D)$ often have some separation properties, whereas, if Φf is integer-valued, $a \equiv b \pmod{I}$ implies $f(a) \equiv f(b) \pmod{I}$ for every ideal I of D.

1.3. **Integer-valued polynomials of a given modulus.** For each nonzero element x of the domain D, we let $\mathrm{Int}_x(E,D)$ be the ring

$$\mathrm{Int}_x(E,D) = \{f \in K[X] \mid \forall a \in E, f(xX + a) \in D[X]\}.$$

Since, for $f \in \mathrm{Int}_x(E,D)$ and $a \in E$, the constant term $f(a)$ of $f(xX+a)$ is in D, the ring $\mathrm{Int}_x(E,D)$ is clearly contained in $\mathrm{Int}(E,D)$. We say that $\mathrm{Int}_x(E,D)$ is the ring of *integer-valued polynomials on E of modulus x*. These rings were introduced by the first author and were studied (for $E = D$) by the third author in her Ph.D. thesis [16], under the advisorship of the second author. Elementary properties can easily be extended to subsets, with essentially the same proofs. We thus recall some of these properties briefly.

1.3.1. If S is a set of representatives of E modulo xD, then

$$\mathrm{Int}_x(E,D) = \bigcap_{a\in S} D\left[\frac{X-a}{x}\right].$$

In particular, $\mathrm{Int}_x(E,D) = \mathrm{Int}_x(S,D)$. If S is finite, E can thus be replaced by a finite set and $\mathrm{Int}_x(E,D)$ can be represented by a finite intersection.

1.3.2. If $x \mid y$ in D, then $\mathrm{Int}_x(E,D) \subset \mathrm{Int}_y(E,D)$.

1.3.3. The rings of modulus x "cover" $\mathrm{Int}(E,D)$:

$$\mathrm{Int}(E,D) = \bigcup_{\substack{x\in D\\ x\neq 0}} \mathrm{Int}_x(E,D).$$

1.4. **Localization properties.** The ring of integer-valued polynomials $\mathrm{Int}(E,D)$ behaves nicely under localization (see [7, I.2]). We briefly show that we have similar properties for $\mathrm{Int}^{\{r\}}(E,D)$ and $\mathrm{Int}_x(E,D)$. These rings are defined by the condition that some polynomials have their values or their coefficients in D. A fortiori, these values or coefficients are then in $S^{-1}D$, for any multiplicative subset S of D. Thus, one always has the containments

$$S^{-1}\mathrm{Int}^{\{r\}}(E,D) \subseteq \mathrm{Int}^{\{r\}}(E,S^{-1}D), \quad S^{-1}\mathrm{Int}_x(E,D) \subseteq \mathrm{Int}_x(E,S^{-1}D).$$

We have equalities under specific hypotheses. Recall that a subset E of K is said to be a *fractional* subset of D if there is a nonzero $d \in D$ such that $dE \subseteq D$.

Proposition 1.4. *Assume E to be a fractional subset of the domain D.*

(1) *If D is Noetherian, then for every multiplicative subset S of D,*

$$S^{-1}\mathrm{Int}^{\{r\}}(E,D) = \mathrm{Int}^{\{r\}}(E,S^{-1}D), \quad S^{-1}\mathrm{Int}_x(E,D) = \mathrm{Int}_x(E,S^{-1}D).$$

(2) *If D is a Krull domain, then for every height one prime ideal $\mathfrak{p}$ of D,*

$$\left(\mathrm{Int}^{\{r\}}(E,D)\right)_{\mathfrak{p}} = \mathrm{Int}^{\{r\}}(E,D_{\mathfrak{p}}), \quad (\mathrm{Int}_x(E,D))_{\mathfrak{p}} = \mathrm{Int}_x(E,D_{\mathfrak{p}}).$$

Proof. As $\mathrm{Int}(E,D)$ and $\mathrm{Int}(dE,D)$ are isomorphic (via mapping X to X/d), we may as well assume that E is a subset of D when studying the ring $\mathrm{Int}(E,D)$ of integer-valued polynomials. The identical argument holds for $\mathrm{Int}^{\{r\}}(E,D)$, while for $\mathrm{Int}_x(E,D)$ we similarly note that $f(X) \in \mathrm{Int}_{dx}(dE,D)$ if and only if $f(X/d) \in \mathrm{Int}_x(E,D)$.

(1) Let $f \in \mathrm{Int}^{\{r\}}(E,S^{-1}D)$ (resp. $f \in \mathrm{Int}_x(E,S^{-1}D)$) and let M be the D-module generated by its coefficients. For $k \leq r$, the values of $\Phi^k f$ (resp. $\forall a \in E$, the coefficients of $f(xX+a)$) are in $M \cap S^{-1}D$. As D is Noetherian, $M \cap S^{-1}D$ is a finitely generated D-module and hence there exists $s \in S$ such that $sf \in \mathrm{Int}^{\{r\}}(E,D)$ (resp. $sf \in \mathrm{Int}_x(E,D)$).

(2) Let $f \in K[X]$. Assuming D to be a Krull domain, there exists $s \in D - \mathfrak{p}$ such that, for each height one prime $\mathfrak{q} \neq \mathfrak{p}$ of D, one has $sf \in D_{\mathfrak{q}}[X]$ [7, proof of Proposition I.2.8]. Thus, for $k \leq r$, the values of $\Phi^k(sf)$ (resp. $\forall a \in E$, the coefficients of $sf(xX+a)$) are in $D_{\mathfrak{q}}$. Now if $f \in \mathrm{Int}^{\{r\}}(E,D_{\mathfrak{p}})$ (resp. $f \in \mathrm{Int}_x(E,D_{\mathfrak{p}})$) these values (resp. these coefficients) are also in $D_{\mathfrak{p}}$, and as $D = \cap_{\mathrm{ht}(\mathfrak{q})=1} D_{\mathfrak{q}}$, we conclude that $sf \in \mathrm{Int}^{\{r\}}(E,D)$ (resp. $sf \in \mathrm{Int}_x(E,D)$). □

Being naturally interested in the classical case where $D = \mathbb{Z}$, or more generally the ring of integers of a number field, these localization properties allow us to focus on the case of a (discrete) valuation domain V, as we next do. Moreover, as we consider integer-valued polynomials on a subset, we impose compactness conditions on this subset and allow V to be non-discrete and of arbitrary dimension, as in [8, 9, 10].

2. v-orderings and regular bases

Hypotheses and notations In this section V denotes a valuation domain, with quotient field K, $\mathfrak{m}$ its maximal ideal, v the corresponding valuation, and Γ its value group. We let $\widehat{K}$ be the completion of K with respect to v. We let E be a subset of K and we assume E to be *precompact*.

We recall that, by definition, E is precompact if its topological closure $\widehat{E}$ in $\widehat{K}$ is compact or equivalently, for each non-zero fractional ideal I of K, E meets finitely many cosets of K modulo I [10, Proposition 1.2]. In particular, a precompact subset is always fractional. A finite subset is clearly precompact, and we recall that if there exists an infinite precompact subset, then K is metrizable [10, Lemma 1.1].

2.1. r-removed v-orderings.

We first study the ring $\mathrm{Int}^{\{r\}}(E,V)$ and give a construction of a regular basis. The key construction is that of an *r-removed v-ordering of E* [5], which is a generalization of the notion of v-ordering introduced by the first author in [3]. Although it was initially considered for subsets of a discrete valuation domain, the v-ordering construction was shown to apply equally well in precompact subsets of valuation domains of dimension 1 [8, 9], or even larger dimension [10]. The same generality holds true for the construction of r-removed v-orderings:

Definition 2.1. An *r-removed v-ordering of E* is a sequence $\{u_n\}$ in E defined inductively in the following way:
— $u_0, u_1, \ldots, u_r$ are arbitrarily chosen;

— for $n > r$, given $u_0, \dots, u_{n-1}$, the next term u_n is chosen so that it minimizes the valuation

$$\sum_{i \in A} v(a - u_i)$$

where a runs through E and A runs through all subsets of $\{0, \dots, n-1\}$ obtained by removing r integers, that is, all subsets containing $n-r$ elements. For $n > r$, the valuation so minimized is denoted by $w_E^{\{r\}}(n)$, and for $n \leq r$, we set $w_E^{\{r\}}(n) = 0$.

Given an r-removed v-ordering of E, we have a subset A_n of $\{0, \dots, n-1\}$ for each n, containing $n - r$ elements, such that, for each other such subset A with $|A| = n - r$ and each $a \in E$, we have

$$w_E^{\{r\}}(n) = \sum_{i \in A_n} v(u_n - u_i) \leq \sum_{i \in A} v(a - u_i).$$

We write $A_n = \{0, \dots, n-1\} \setminus \{n_1, \dots, n_r\}$.

The existence of an r-removed v-ordering follows immediately from the fact that E is precompact [10, Corollary 1.6] (and is obvious if the valuation is discrete) but, as the initial terms may be arbitrarily chosen, it is clearly not unique. However, it turns out that $w_E^{\{r\}}(n)$ is independent from the chosen r-removed v-ordering, just like for ordinary v-orderings [3], as follows directly from the link described below with regular bases.

Remarks 2.2. (1) In analogy with generalized factorials [3], we may let $n!_E^{\{r\}}$ be the ideal formed by the elements x of V such that $v(x) \geq w_E^{\{r\}}(n)$. This is a principal ideal and every element with valuation $w_E^{\{r\}}(n)$ is a generator, as is for instance $\prod_{i \in A_n}(u_n - u_i)$. These ideals are termed the *r-removed factorials* of E in [5].

(2) An element $\alpha \in E$ may occur $r + 1$ times in an r-removed v-ordering: for instance we may choose $u_0 = u_1 = \ldots = u_r = \alpha$. If E is infinite, a given element may occur at most $r + 1$ times, and $w_E^{\{r\}}(n) < \infty$ for all n. If E is finite, some element is bound to occur more than $r + 1$ times and $w_E^{\{r\}}(n) = \infty$ for n large enough.

Associated regular bases. Just as a v-ordering can be used to provide a basis of $\mathrm{Int}(E, V)$ [3], [9, 10], a regular basis of $\mathrm{Int}^{\{r\}}(E, V)$ is associated to an *r-removed v-ordering* $\{u_n\}$. For simplicity, we assume E to be a subset of V.

We set $\binom{X}{0}_E^{\{r\}} = 1$. For $n > 0$, we choose $d_n \in V$ such that $v(d_n) = w_E^{\{r\}}(n)$, and let $\binom{X}{n}_E^{\{r\}}$ be the degree n polynomial:

$$\binom{X}{n}_E^{\{r\}} = \frac{1}{d_n} \prod_{i=0}^{n-1} (X - u_i).$$

For $n \leq r$, one can choose $d_n = 1$, so $\binom{X}{n}_E^{\{r\}} = (X - u_0)(X - u_1)...(X - u_{n-1})$. For $n > r$, one can choose $d_n = \prod_{i \in A_n}(u_n - u_i)$.

With these notations, we have the following [5]:

Proposition 2.3. *The polynomials $\binom{X}{n}_E^{\{r\}}$ form a regular basis of* $\mathrm{Int}^{\{r\}}(E, V)$.

From Proposition 2.3, it follows in particular that $w_E^{\{r\}}(n)$ is independent of the v-ordering $\{u_n\}$.

2.2. **v-ordering of order α.** We now turn to describing a regular basis for $\mathrm{Int}_x(E,V)$, which will be important for us in describing its Noetherian properties. It follows from 1.3.2 that $\mathrm{Int}_x(E,V)$ depends only on the valuation $\alpha = v(x)$. The key notion in describing a regular basis for $\mathrm{Int}_x(E,V)$ is that of a *v-ordering of order* α [5]; it was studied in detail in the case $E = V$ in [16]:

Definition 2.4. A *v-ordering of order α of E* is a sequence $\{u_n\}$ in E defined inductively in the following way:
— u_0 is arbitrarily chosen;
— for $n \geq 1$, given $u_0, u_1, ..., u_{n-1}$, the next term u_n is chosen such that it minimizes the valuation

$$\sum_{i<n} \inf(\alpha, v(a-u_i))$$

where a runs through E. We set $w_E^{(\alpha)}(0) = 0$ and for $n > 0$, denote by $w_E^{(\alpha)}(n)$ this minimized valuation: $w_E^{(\alpha)}(n) = \sum_{i<n} \inf(\alpha, v(u_n - u_i))$.

Given a v-ordering of order α, we have for each $a \in E$ that

$$w_E^{(\alpha)}(n) \leq \sum_{i<n} \inf(\alpha, v(a-u_i)).$$

As with r-removed v-orderings, the existence of v-orderings of order α follows immediately from the fact that E is precompact.

Associated regular bases. Just as an r-removed v-ordering of E gives a basis of $\mathrm{Int}^{\{r\}}(E,V)$, a regular basis of $\mathrm{Int}_x(E,V)$ can be obtained using a v-ordering of order α. Again, let us assume E to be a subset of V. Set $\binom{X}{0}_{E,\alpha} = 1$. For $n > 0$, we choose $d_n \in V$ such that $v(d_n) = w_E^{(\alpha)}(n)$, and let $\binom{X}{n}_{E,\alpha}$ be the degree n polynomial:

$$\binom{X}{n}_{E,\alpha} = \frac{1}{d_n} \prod_{i=0}^{n-1} (X - u_i).$$

With these notations, we have the following [5], [16]:

Proposition 2.5. *The polynomials $f_n = \binom{X}{n}_{E,\alpha}$ form a regular basis of* $\mathrm{Int}_x(E,V)$.

It follows immediately that $w_E^{(\alpha)}(n)$ is independent of the v-ordering $\{u_n\}$.

Further properties of v-orderings of order α. We collect here a few easy lemmas on v-orderings of order α that we will need, extending in some cases the results obtained in [16] in the particular case where $E = V$ and the valuation is discrete.

Lemma 2.6. *The sequence $\{w_E^{(\alpha)}(n)\}$ (with values in the value group Γ of v) is non-decreasing. In fact,*

$$w_E^{(\alpha)}(n) \leq w_E^{(\alpha)}(n+1) \leq w_E^{(\alpha)}(n) + \alpha.$$

Proof. Indeed, on the one hand

$$\begin{aligned} w_E^{(\alpha)}(n+1) &= \sum_{i<n+1} \inf(\alpha, v(u_{n+1}-u_i)) \geq \sum_{i<n} \inf(\alpha, v(u_{n+1}-u_i)) \\ &\geq \sum_{i<n} \inf(\alpha, v(u_n-u_i)) = w_E^{(\alpha)}(n). \end{aligned}$$

And on the other,

$$\begin{aligned} w_E^{(\alpha)}(n+1) &= \sum_{i<n+1} \inf(\alpha, v(u_{n+1}-u_i)) \leq \sum_{i<n+1} \inf(\alpha, v(u_n-u_i)) \\ &\leq \sum_{i<n} \inf(\alpha, v(u_n-u_i)) + \inf(\alpha, v(0)) = w_E^{(\alpha)}(n) + \alpha. \end{aligned}$$

□

Let I_α denote the ideal $I_\alpha = \{x \in V \mid v(x) \geq \alpha\}$. As E is precompact, it meets finitely many classes modulo I_α. If q_α is the number of these classes, then it follows from Lemma 2.6 that

$$\left[\frac{n}{q_\alpha}\right] \alpha \leq w_E^{(\alpha)}(n) \leq n\alpha, \tag{7}$$

and $\inf(\alpha, v(a-u_i)) = \alpha$ for $a \equiv u_i \pmod{I_\alpha}$.

Remark 2.7. The sum $\sum \inf(\alpha, v(a-u_i))$ is unchanged when a is replaced by any element equivalent modulo I_α. Indeed, we already noticed that we can replace E by the finite set formed by a set of representatives modulo I_α.

Lemma 2.8. *Given a sequence $\{u_n\}$, set $w(n) = \sum_{i<n} \inf(\alpha, v(u_n-u_i))$. Then $\{u_n\}$ is a v-ordering of order α if and only if $w(n) = w_E^{(\alpha)}(n)$ for each n.*

Proof. The condition is clearly necessary. Conversely, suppose $\{u_n\}$ is not a v-ordering of order α, and let n be the smallest integer such that u_n does not minimize the sum $\sum_{i<n} \inf(\alpha, v(a-u_i))$. Then there is a v-ordering of order α starting with $u_0, \ldots, u_{n-1}$ and followed with $u'_n \neq u_n$. But then,

$$w_E^{(\alpha)}(n) = \sum_{i<n} \inf(\alpha, v(u'_n-u_i)) < \sum_{i<n} \inf(\alpha, v(u_n-u_i)) = w(n).$$

□

Finally, if $\{u_n\}$ is a sequence in E and $x, y \in V$, then $\{xu_n + y\}$ is a sequence in $E' = xE + y$. We have:

Lemma 2.9. *Let $\{u_n\}$ be a sequence in E and $x, y \in V$, with $v(x) = \beta$. Then $\{u_n\}$ is a v-ordering of order α in E if and only if $\{xu_n + y\})$ is a v-ordering of order $\alpha + \beta$ in $E' = xE + y$. Moreover, for each n, we have*

$$w_{E'}^{(\alpha+\beta)}(n) = w_E^{(\alpha)}(n) + n\beta.$$

Proof. For $a \in E$, write $a' = xa + y$. Then $a' \in E'$ and $a' - u'_n = x(a-u_n)$. Hence

$$\sum_{i<n} \inf(\alpha+\beta, v(a'-u'_i)) = \sum_{i<n} \inf(\alpha+\beta, v(a-u_i)+\beta) = \sum_{i<n} \inf(\alpha, v(a-u_i)) + n\beta.$$

□

Since E is precompact, it meets finitely many classes modulo the maximal ideal $\mathfrak{m}$; we denote these classes by $E_0, E_1, ..., E_{q-1}$.

Lemma 2.10. *Let $\{u_n\}$ be a sequence in E. Set $w(n) = \sum_{i<n} \inf(\alpha, v(u_n - u_i))$. Then $\{u_n\}$ is a v-ordering of order α of E if and only if*

(1) *The sequence $\{w(n)\}$ is non-decreasing,*

(2) *for each class E_j of E modulo $\mathfrak{m}$, the terms of $\{u_n\}$ in E_j form an infinite subsequence $\{u_n^{(j)}\}$ which is a v-ordering of order α of E_j.*

Proof. • Assume that $\{u_n\}$ is a v-ordering sequence of order α in S. Then (1) follows from Lemma 2.6. As for (2), consider a class E_j of E modulo $\mathfrak{m}$. By way of contradiction, suppose there is an integer N such that, for $n \geq N$, $u_n \notin E_j$. Then, for $a \in E_j$ and $n \geq N$,

$$\sum_{i<n} \inf(\alpha, v(a - u_i)) = \sum_{i<N} \inf(\alpha, v(a - u_i)) \leq N\alpha.$$

If $\{u_n\}$ is a v-ordering, then $w(n) = w_E^{(\alpha)}(n) \geq \left[\frac{n}{q_\alpha}\right] \alpha$ by (7) and Lemma 2.8. But, for n large, $\left[\frac{n}{q_\alpha}\right] \alpha \geq N\alpha$, and so $\{u_n\}$ would not be a v-ordering. We conclude that the terms of $\{u_n\}$ in E_j form an infinite subsequence $\{u_n^{(j)}\}$.

Now consider an element $u_m^{(j)}$ of this subsequence. Let n be the integer such that $u_m^{(j)} = u_n$. Since, for $a \in E_j$, $v(a - u_i) = 0$ if $u_i \notin E_j$, one has

$$\sum_{i<n} \inf(\alpha, v(a - u_i)) = \sum_{i<n, u_i \in E_j} \inf(\alpha, v(a - u_i)) = \sum_{k<m} \inf(\alpha, v(a - u_k^{(j)})).$$

As a runs through E, the left hand side is minimal for $a = u_n$, that is, for $a = u_m^{(j)}$. Thus the right hand side is also minimal for $a = u_m^{(j)}$ as a runs through E_j, implying that $u_m^{(j)}$ is the m-th term in a v-ordering of order α of E_j.

• Conversely, assume (1) and (2). We show that, for each integer $n \geq 0$ and each $a \in E$,

$$\sum_{i<n} \inf(\alpha, v(a - u_i)) \geq \sum_{i<n} \inf(\alpha, v(u_n - u_i)).$$

Denoting by E_j the class of u_n modulo $\mathfrak{m}$, we consider two cases:

(i) $a \in E_j$. As above,

$$\sum_{i<n} \inf(\alpha, v(a - u_i)) = \sum_{i<n, u_i \in E_j} \inf(\alpha, v(a - u_i)).$$

As the sequence formed by the elements of $\{u_n\}$ in E_j is a v-ordering of E_j, this sum is minimal (as a runs through E_j) for $a = u_n$.

(ii) $a \notin E_j$. Let n' be the least integer, $n' > n$, such that $a \equiv u'_n \pmod{\mathfrak{m}}$. Then, $v(a - u_i) = 0$, for $n \leq i < n'$, and

$$\sum_{i<n} \inf(\alpha, v(a - u_i)) = \sum_{i<n'} \inf(\alpha, v(a - u_i)).$$

From case (i) above,

$$\sum_{i<n'} \inf(\alpha, v(a - u_i)) \geq \sum_{i<n'} \inf(\alpha, v(u'_n - u_i)) = w(n').$$

As the sequence $\{w(n)\}$ is non-decreasing, we have in conclusion

$$\sum_{i<n} \inf(\alpha, v(a-u_i)) \geq w(n') \geq w(n) = \sum_{i<n} \inf(\alpha, v(u_n - u_i)).$$

□

Now given v-orderings $\{u_n^{(j)}\}$ of order α for each class E_j, we reciprocally can construct a v-ordering $\{u_n\}$ of order α for E. For simplicity we denote $w_{E_j}^{(\alpha)}(n)$ by $w_j(n)$, that is, $w_j(m) = \sum_{i<m} \inf(\alpha, v(u_m^{(j)} - u_i^{(j)}))$. We observe that, if $\alpha > 0$, then the sequence $\{w_j(n)\}$ is strictly increasing. Indeed,

$$w_j(m+1) = \inf(\alpha, v(u_{m+1}^{(j)} - u_m^{(j)})) + \sum_{i<m} \inf(\alpha, v(u_{m+1}^{(j)} - u_i^{(j)})).$$

On the one hand, $v(u_{m+1}^{(j)} - u_m^{(j)}) > 0$, since the elements of the sequence $\{u_n^{(j)}\}$ are in the same class modulo $\mathfrak{m}$, and on the other hand,

$$\sum_{i<m} \inf(\alpha, v(u_{m+1}^{(j)} - u_i^{(j)})) \geq \sum_{i<m} \inf(\alpha, v(u_m^{(j)} - u_i^{(j)})) = w_j(m).$$

We can merge the strictly increasing sequences $\{w_j(n)\}$ into a single nondecreasing sequence (w_n), and then merge the q sequences $\{u_n^{(j)}\}$ into a sequence $\{u_n\}$ following the same order: if the n-th term of the sequence (w_n) is $w_j(m)$, then $u_n = u_m^{(j)}$. It follows from Lemma 2.10 that $\{u_n\}$ is a v-ordering of order α.

Remark 2.11. Although we shall not need it here, we note that the natural analogue of Lemma 2.10 also holds for r-removed v-orderings.

3. Finiteness properties

We are now ready to address the question of the Noetherian properties of both rings $\mathrm{Int}_x(E, D)$ and $\mathrm{Int}^{\{r\}}(E, D)$, where D is a Dedekind domain and E a locally precompact fractional subset of D. We first turn to $\mathrm{Int}_x(E, D)$, and show that it is in fact finitely generated as a D-module. As we will see, the key to obtaining this result is the existence of a *periodic* v-ordering of order $\alpha = v(x)$, which we prove in Theorem 3.1 below. If D is replaced by a general valuation domain (not necessarily discrete), then such a periodic v-ordering need not exist for E (see Example 3.5 below), and indeed we show that $\mathrm{Int}_x(E, D)$ may not be finitely generated in this case.

In contrast, we demonstrate in Sections 3.2–3.3 that $\mathrm{Int}^{\{r\}}(E, D)$ is never Noetherian for any Dedekind domain D, and neither is $\mathrm{Int}^{\{r\}}(E, V)$ for any valuation domain V. We prove this using a “separation argument” which, in particular, allows us to construct specific ideals that are not finitely generated.

Nevertheless, we show in Section 3.4 that all these rings satisfy the “ascending chain condition on principal ideals”. In the final Section 3.5, we discuss how all the aforementioned results generalize also to the combined ring $\mathrm{Int}_x^{\{r\}}(E, D)$.

3.1. **Finite generation of** $\mathrm{Int}_x(E, D)$**.** We begin by examining the case of $\mathrm{Int}_x(E, V)$, where V is a discrete valuation domain. In this case, we find that E must possess a periodic v-ordering of order α for any integer α:

Theorem 3.1. *Let V be a discrete valuation domain, and let E be a precompact subset of the quotient field K of V. Then, for any integer α, there exists a periodic v-ordering of order α in E. If the period is θ, then for all positive integers k and n, we have*

$$w_E^{(\alpha)}(k\theta + n) = kw_E^{(\alpha)}(\theta) + w_E^{(\alpha)}(n).$$

Proof. We first observe that, for the last statement, it is clearly enough to prove that, for each n,

$$w_E^{(\alpha)}(\theta + n) = w_E^{(\alpha)}(\theta) + w_E^{(\alpha)}(n).$$

By replacing E by xE for some $x \in V$ if necessary, we may assume $E \subset D$. We prove Theorem 3.1 by induction on α. If $\alpha \leq 0$, then any sequence $\{u_n\}$ of elements from E is a v-ordering of order α. In particular, any constant sequence is a periodic v-ordering of order α; as moreover $w_E^{(\alpha)}(n) = 0$ for each n, the result is true in this case.

Now supposing $\alpha > 0$, and the result true for v-orderings of order less than α, we let $E_1, ..., E_s$ denote the classes of E modulo $\mathfrak{m}^\alpha$, where $\mathfrak{m}$ denotes the maximal ideal of V. From Lemma 2.9, for each j and each $a \in E_j$, the v-orderings of order α in E_j are of the form $(tv_n + a)$ where $\{v_n\}$ is a v-ordering of order $\alpha - 1$ in $E'_j = \{x \in V \mid tx + a \in E_j\}$.

By the induction hypothesis there exists a periodic v-ordering of order $\alpha - 1$ on each E'_j, and hence, a periodic v-ordering $\{u_n^{(j)}\}$ of order α on each E_j. We denote by θ_j the period of $\{u_n^{(j)}\}$ and, as above, write $w_j(n)$ for $w_{E_j}^{(\alpha)}(n)$. Then $w_j(\theta_j + n) = w_j(\theta_j) + w_j(n)$, for each n. Indeed, from Lemma 2.9, we have

$$w_j(\theta_j + n) = w_{E_j}^{(\alpha)}(\theta_j + n) = w_{E'_j}^{(\alpha-1)}(\theta_j + n) + \theta_j + n.$$

But, from the induction hypothesis,

$$w_{E'_j}^{(\alpha-1)}(\theta_j + n) = w_{E'_j}^{(\alpha-1)}(\theta_j) + w_{E'_j}^{(\alpha-1)}(n).$$

Thus

$$w_j(\theta_j + n) = w_{E'_j}^{(\alpha-1)}(\theta_j) + \theta_j + w_{E'_j}^{(\alpha-1)}(n) + n = w_j(\theta_j) + w_j(n).$$

Let $w = \operatorname{lcm}_{1\leq j\leq s}(w_j(\theta_j))$, $\lambda_j = \dfrac{w}{w_j(\theta_j)}$, and $\theta = \sum_{1\leq j\leq s} \lambda_j\theta_j$. As above, it follows from Lemma 2.10 that we obtain a v-ordering $\{u_n\}$ of order α by merging the sequences $\{u_n^{(j)}\}$. Before choosing a term corresponding to the value w, one must choose all the terms corresponding to a smaller value, and hence the first θ terms of the sequence $\{u_n\}$ are the union of the first $\lambda_j\theta_j$ terms of each sequence $\{u_n^{(j)}\}$. As each sequence $\{u_n^{(j)}\}$ is periodic, the v-ordering $\{u_n\}$ we obtain is thus periodic of period θ. For each j, we then have

$$w(\theta) = w = \lambda_j w_j(\theta_j) = w_j(\lambda_j\theta_j).$$

Finally, if u_n is in E_j, then so is $u_{n+\theta}$. More precisely, if we write $u_n = u_m^{(j)}$, then $u_{n+\theta} = u_{m+\lambda_j\theta_j}^{(j)}$. It follows that

$$w(n + \theta) = w_j(m + \lambda_j\theta_j) = w_j(m) + w_j(\lambda_j\theta_j) = w(n) + w(\theta),$$

as desired. □

Since the valuation on V is discrete, the regular basis associated to the v-ordering $\{u_n\}$ of order α [Proposition 2.5] can be taken to be of the form $f_n = \frac{(X-u_0)(X-u_1)...(X-u_{n-1})}{t^{w_S^{(\alpha)}(n)}}$, where t denotes a uniformizer for V, that is, an element such that $v(t) = 1$. If we choose $\{u_n\}$ to be θ-periodic (Theorem 3.1) and let $n = k\theta + r$ with $0 \leq r < \theta$, then $w_S^{(\alpha)}(n) = kw_S^{(\alpha)}(\theta) + w_S^{(\alpha)}(r)$, and thus

$$f_n = \frac{[(X-u_0)(X-u_1)\dots(X-u_{\theta-1})]^k}{t^{kw_S^{(\alpha)}(\theta)}} \frac{(X-u_\theta)\dots(X-u_{n-1})}{t^{w_S^{(\alpha)}(r)}} = f_\theta^k f_r.$$

Hence we obtain the following:

Theorem 3.2. *Let V be a discrete valuation domain with quotient field K, E a precompact subset of K, and x any non-zero element of V of valuation α. Then* $\mathrm{Int}_x(E, V)$ *is a finitely generated V-algebra. More precisely, if $\{u_n\}$ is a θ-periodic v-ordering of order α on E and $\{f_n\}$ the corresponding regular basis, then*

$$\mathrm{Int}(E, V) = V[f_1, ...f_\theta].$$

In particular, it follows that $\mathrm{Int}_x(E, V)$ is Noetherian.

By globalization, we can extend this result to Krull domains. We consider a subset E which is locally precompact, that is, is precompact in $D_\mathfrak{p}$ for each height one prime $\mathfrak{p}$.

Corollary 3.3. *Let D be a Krull domain, x a nonzero element of D, and E a locally precompact fractional subset of D. Then* $\mathrm{Int}_x(E, D)$ *is a finitely generated D-algebra.*

Proof. Since D is a Krull domain, we have

$$\mathrm{Int}_x(E, D) = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} \mathrm{Int}_x(E, D_\mathfrak{p}) = [\mathrm{Int}(E, D)]_\mathfrak{p}$$

by Proposition 1.4 (2). Now, x is a unit in each $D_\mathfrak{p}$ except for some finite set Q of height one primes.

— For $\mathfrak{p} \notin Q$, x is invertible in $D_\mathfrak{p}$, and $\mathrm{Int}_x(E, D_\mathfrak{p}) = D_\mathfrak{p}[X]$.

— For $\mathfrak{p} \in Q$, $\mathrm{Int}_x(E, D_\mathfrak{p})$ is a finitely generated $D_\mathfrak{p}$-algebra by Proposition 3.2, and we denote the finite set of generators by $Y_\mathfrak{p}$. Note that each generator being a priori in $[\mathrm{Int}(E, D)]_\mathfrak{p}$ can, in fact, be taken in $\mathrm{Int}_x(E, D)$.

We set $Y = \bigcup_{\mathfrak{p}\in Q} Y_\mathfrak{p}$. Then Y is a finite set of generators of the D-algebra $\mathrm{Int}_x(E, D)$. □

Corollary 3.4. *Let D be Noetherian integrally closed domain, x a non-zero element of D, and E a locally precompact fractional subset of D. Then* $\mathrm{Int}_x(E, D)$ *is a finitely generated D-algebra and is Noetherian.*

In particular, if D is the ring of integers in a number field K and E is a fractional subset of K, then $\mathrm{Int}_x(E, D)$ is Noetherian, as every fractional subset is locally precompact.

Let us now turn to a general valuation domain V, and consider a precompact subset E of V and an element α in the value group of V. In this case, contrary to Theorem 3.1, the set E may fail to have a periodic v-ordering of order α and then, contrary to Theorem 3.2, $\mathrm{Int}_x(E, V)$ may not even be finitely generated. This is illustrated in the following example.

Example 3.5. Let V be a valuation domain with value group the real numbers, and let $E = \{1, x, y+1\}$ where $x, y \in V$ are such that $v(x) = \pi$ and $v(y) = 1$. Then E does not possess a periodic v-ordering of order π and $\mathrm{Int}_x(E, V)$ is not finitely generated.

Indeed, modulo the maximal ideal $\mathfrak{m}$, there are two residue classes: $E_1 = \{x\}$ and $E_2 = \{1, y+1\}$. Each has a periodic v-ordering, namely $x, x, x, x, x, \ldots$ and $1, y+1, 1, y+1, 1, \ldots$ respectively. Their respective w-sequences are $0, \pi, 2\pi, 3\pi, 4\pi, \ldots$ and $0, 1, 1+\pi, 2+\pi, 2+2\pi, 3+2\pi, \ldots$.

When these two v-orderings are merged (via merging the w-sequences) into a v-ordering of E, it cannot happen in a periodic way. If it did, then up to a large number N that is a multiple of the period, each sequence would have to occur with a relative frequency of exactly $1/\pi$ and $2/(1+\pi)$ respectively in the merged sequence. But the ratio of these two frequencies is irrational, so this cannot occur. (The proof of Theorem 3.1 does not apply here because π and $(1+\pi)/2$ do not have any "common multiple", since they are not rationally related.)

To see that $\mathrm{Int}_x(E, V)$ is not finitely generated, let $\{u_n\}$ denote the v-ordering of order π obtained by merging the v-orderings of E_1 and E_2, and let $k_0, k_1, k_2, \ldots$ denote the (strictly increasing) sequence of those integers for which $u_{k_i} = x$. Let $f_n(X) = \frac{(X-u_0)(X-u_1)\ldots(X-u_{n-1})}{d_n}$ denote the corresponding regular basis of $\mathrm{Int}_x(E, V)$. Since each $f_n(X) \in \mathrm{Int}_x(E, V)$, we have in particular $f_n(xX + x) \in V[X]$. In fact, it is easy to see from the definition of f_n (modifying d_n by a unit factor in V if necessary) that modulo $\mathfrak{m}$ we have

$$f_n(xX + x) \equiv \begin{cases} X^i \pmod{\mathfrak{m}} & \text{if } n = k_i \\ 0 \pmod{\mathfrak{m}} & \text{otherwise.} \end{cases} \tag{8}$$

From (8) it follows that k_i is the minimum degree of any polynomial $f \in \mathrm{Int}_x(E, V)$ satisfying $f(xX + x) \equiv X^i \pmod{\mathfrak{m}}$. Furthermore, since the frequency of the occurrence of x in the v-ordering $\{u_n\}$ is $(1+\pi)/(1+3\pi)$, we have

$$\lim_{i\to\infty} \frac{i}{k_i} = \frac{1+\pi}{1+3\pi}. \tag{9}$$

Now suppose $\mathrm{Int}_x(E, V)$ is finitely generated, that is, suppose it is of the form $R_m = V[f_0, \ldots, f_m]$ for some m. (Since the f_i form a regular basis of $\mathrm{Int}_x(E, V)$, any finitely generated subring of $\mathrm{Int}_x(E, V)$ must be contained in R_m for some m.) Let k'_i denote the minimal degree of any polynomial $f \in R_m$ such that $f(xX+x) \equiv X^i \pmod{\mathfrak{m}}$. If $R_m = \mathrm{Int}_x(E, V)$, then we must have $k'_i = k_i$. However, it is also follows from (8) and the definition of R_m that

$$\limsup_{i\to\infty} \frac{i}{k'_i} = \max_{\{i: 0 \le k_i \le m\}} \frac{i}{k_i} \tag{10}$$

which is a rational number, being the maximum of a finite set of rational numbers. This contradicts (9), and thus $\mathrm{Int}_x(E, V)$ is not finitely generated.

In conclusion, we have shown that $\mathrm{Int}_x(E, V)$ is Noetherian (for E a precompact set in the quotient field K of V) whenever V is a discrete valuation domain, while if V is a domain with a non-discrete valuation, then the finite generation of $\mathrm{Int}_x(E, V)$ in general depends on the structure of E and the value group Γ of V. If common integer multiples exist in Γ (e.g., they always do for $\Gamma = \mathbb{Z}$ or even $\Gamma = \mathbb{Q}$), then $\mathrm{Int}_x(E, V)$ is always finitely generated over V for precompact subsets E.

3.2. **Separation of points.** We now turn our attention to the ring $\mathrm{Int}^{\{r\}}(E, D)$, beginning with the case where $D = V$ is a valuation domain. We assume first the existence of cluster points, that is, that the subset E is infinite.

As in any metric space, we say that $\alpha \in K$ is a *cluster point of the subset* E if there are elements of E, distinct from α, that are arbitrarily close to α; that is,

$$\forall\gamma \in \Gamma, \exists a \in E, a \neq \alpha, \text{ with } v(a - \alpha) \geq \gamma.$$

If E admits a cluster point, it must then be infinite: for each $\gamma \in \Gamma$, there are infinitely many elements $a \in E$ such that $v(a - \alpha) \geq \gamma$.

Similarly, we say that α is a *cluster point of a sequence* $\{u_n\}$ if

$$\forall\gamma \in \Gamma, \forall N, \exists n > N, \text{with } v(u_n - \alpha) > \gamma.$$

Proposition 3.6. *Let $\{u_n\}$ be an r-removed v-ordering of E. If α is a cluster point of E, then it is also a cluster point of any r-removed v-ordering $\{u_n\}$ of E.*

Proof. By way of contradiction, we suppose that

$$\exists\gamma \in \Gamma, \exists N, n > N \Longrightarrow v(u_n - \alpha) < \gamma. \tag{11}$$

We consider the ideal $I_\gamma = \{x \in V \mid v(x) \geq \gamma\}$. As E is precompact, it meets finitely many classes modulo I_γ. We choose a set of representatives $\{\alpha_0, \alpha_1, \ldots, \alpha_s\}$, with $\alpha_0 = \alpha$, and denote by $[\alpha_i]$ the class of α_i. Thus, x, y are in the same class if and only if $v(x - y) \geq \gamma$. On the other hand, if x, y lie in distinct classes, that is, $x \in [\alpha_i], y \in [\alpha_j],\ i \neq j$, then we have

$$v(x - y) = v(\alpha_i - \alpha_j) < \gamma. \tag{12}$$

We say a class is *infinite* if it contains infinitely many terms of the sequence $\{u_n\}$, otherwise we say it is *finite*. There are classes of both kinds; indeed, the class $[\alpha_0]$ of $\alpha = \alpha_0$ is finite from (11) and, on the other hand, as there are finitely many classes but the sequence $\{u_n\}$ is infinite, there exists at least one infinite class. Renumbering the classes if need be, we let $[\alpha_1]$ be such that, among the infinite classes, $v(\alpha_0 - \alpha_1)$ is maximal. For another infinite class $[\alpha_j]$, we thus have

$$v(\alpha_1 - \alpha_j) \geq \inf\{v(\alpha_1 - \alpha_0), v(\alpha_0 - \alpha_j)\} = v(\alpha_0 - \alpha_j) \tag{13}$$

We choose an element b in $[\alpha_0]$ which is not a term of the sequence $\{u_n\}$ (this is possible since $\alpha = \alpha_0$ is a cluster point of E and there are only finitely many terms of the sequence $\{u_n\}$ in $[\alpha_0]$).

We claim there is an integer M such that,

$$u_n \in [\alpha_1], i > M \Longrightarrow v(u_n - u_i) \geq v(b - u_i). \tag{14}$$

For this, it is enough to show that, for $u_n \in [\alpha_1]$ and u_i in any infinite class, we have $v(u_n - u_i) \geq v(b - u_i)$ and we consider two cases:

- $u_i \in [\alpha_1]$. Then $v(u_n - u_i) \geq \gamma > v(b - u_i)$. Indeed, u_n and u_i are in the same class modulo I_γ while b and u_i are not.
- $u_i \in [\alpha_j]$ for some infinite class $[\alpha_j] \neq [\alpha_1]$. Then, from (12) and (13),

$$v(u_n - u_i) = v(\alpha_1 - \alpha_j) \geq v(\alpha_0 - \alpha_j) = v(b - u_i).$$

We set $S = \sum_{i=0}^{M} v(b - u_i)$. As the class $[\alpha_1]$ is infinite and E is precompact, we can choose $u_n \in [\alpha_1]$ with at least $r + 1$ terms $u_i, i < n$, with $v(u_n - u_i) > S + \gamma$.

Hence, if A is a subset of the interval $\{0,\ldots,n-1\}$ obtained by removing r integers, there remains $j \in A$ such that

$$v(u_n - u_j) > S + \gamma > \sum_{i=0}^{M} v(b-u_i) + v(b-u_j). \tag{15}$$

Writing

$$\sum_{i\in A} v(u_n - u_i) = v(u_n - u_j) + \sum_{i\in A, i\neq j} v(u_n - u_i),$$

we conclude from (14) and (15) that we have

$$\sum_{i\in A} v(u_n - u_i) > \sum_{i=0}^{M} v(b-u_i) + \sum_{i\in A, i>M} v(b-u_i) \geq \sum_{i\in A} v(b-u_i).$$

We thus reach a contradiction with the definition of an r-removed v-ordering. □

As said in the introduction, the fact that $\mathrm{Int}(V)$ is not Noetherian follows from a property of separation of points [7, §III.4]): for $a \neq b$ in V, there exists $f \in \mathrm{Int}(V)$ with $f(a) \equiv 0 \pmod{\mathfrak{m}}$ and $f(b) \equiv 1 \pmod{\mathfrak{m}}$. This does not apply to divided differences: if $\frac{f(Y)-f(X)}{Y-X}$ is integer valued, then $a \equiv b \pmod{\mathfrak{m}}$ implies $f(a) \equiv f(b)$ $\pmod{\mathfrak{m}}$. We shall nevertheless use a separation of points argument to prove that $\mathrm{Int}^{\{r\}}(E,V)$ is not Noetherian.

Replacing K by its completion $\widehat{K}$, then $\widehat{E}$ is compact. If E is infinite, then $\widehat{E}$ admits a cluster point α.

Lemma 3.7. *Let α be a cluster point of E in $\widehat{E}$. For each neighborhood U of α in $\widehat{E}$, one can find $\beta \in U$, and a polynomial $f \in \mathrm{Int}^{\{r\}}(\widehat{E},\widehat{V})$ with*

$$\begin{cases} f(\alpha) = 0 \\ \Phi f(\alpha,\alpha) = 0 \\ \ldots \\ \Phi^r f(\alpha,\ldots,\alpha,\alpha) = 0 \\ \Phi^r f(\alpha,\ldots,\alpha,\beta) = 1. \end{cases}$$

Proof. We let $\{u_n\}$ be an r-removed v-ordering of $\widehat{E}$ and $\{f_n\}$ be the corresponding basis of $\mathrm{Int}^{\{r\}}(\widehat{E},\widehat{V})$. The element β we are looking for is the term u_n of this ordering, for some $n > r$, and the polynomial f the corresponding f_n. We write $f_n = \frac{\prod_{k=0}^{n-1}(X-u_k)}{d_n}$ (and explicit d_n only when need be). As the first $r+1$ terms of a r-removed v-ordering are arbitrarily chosen, we can let $u_0 = \ldots = u_r = \alpha$ (and as E is infinite, we then have $u_i \neq \alpha$ for $i > r$ [Remark 2.2 (2)]). For $n > r$, $(X-\alpha)^{r+1}$ is then a factor of f_n. It then follows from the relation $f^{(k)}(X) = k!\Phi^k f(X,\ldots,X)$ [(1.1.7)], that

$$f_n(\alpha) = 0,\ \Phi f_n(\alpha,\alpha) = 0,\ \ldots,\ \Phi^r f_n(\alpha,\ldots,\alpha,\alpha) = 0. \tag{16}$$

On the other hand, recalling formula 1.1.5:

$$\begin{aligned} f_n(X) \quad &= \quad f_n(a) + (X-a)\Phi^1 f_n(a,b) + (X-a)(X-b)\Phi^2 f_n(a,b,c) + \ldots \\ &\quad \ldots + (X-a)(X-b)\ldots(X-h)\Phi^r f_n(a,b,c,\ldots,h,X), \end{aligned}$$

and making $a = b = \ldots = h = \alpha$, and $X = \beta$, it follows from (16) that we have

$$\Phi^r f_n(\alpha, \ldots, \alpha, \beta) = \frac{1}{d_n} \prod_{i=r}^{n-1} (\beta - u_i). \tag{17}$$

As α is a cluster point of the sequence $\{u_n\}$ by Proposition 3.6, there is $n > r$ such that $u_n \in U$ satisfies the inequalities $v(u_n - \alpha) > v(u_i - \alpha)$ for $r < i < n$, and thus $v(u_n - \alpha) > v(u_n - u_i)$. We then choose $\beta = u_n$ and $f = f_n$. It follows from these choices that the valuation of the product $\prod_{i \in A}(u_n - u_i)$ (where $A \subset \{0, \ldots, n-1\}$ is obtained by removing r integers) is minimal when one removes the first r factors (for which $u_i = \alpha$ and $v(u_n - u_i)$ is maximal). The denominator d_n of f_n can thus be written as the product $\prod_{i=r}^{n-1}(u_n - u_i) = \prod_{i=r}^{n-1}(\beta - u_i)$. From (17), we thus obtain

$$\Phi^r f_n(\alpha, \ldots, \alpha, \beta) = \frac{1}{d_n} \prod_{i=r}^{n-1} (\beta - u_i) = 1.$$

□

As E is topologically dense in $\widehat{E}$, polynomials that are integer-valued on E are, by continuity, integer-valued on $\widehat{E}$. Thus $\mathrm{Int}^{\{r\}}(E, V)$ is contained in and even dense in $\mathrm{Int}^{\{r\}}(\widehat{E}, \widehat{V})$ for the uniform convergence topology. Setting $\widehat{\mathfrak{m}}$ to be the maximal ideal of $\widehat{V}$ (or, equivalently, the closure of the maximal ideal $\mathfrak{m}$ of V), we conclude with the following property of separation of points by finite differences.

Proposition 3.8. *Let α be a cluster point of E in $\widehat{E}$. Then, for each neighborhood U of α in $\widehat{E}$, one can find $\beta \in U$, and a polynomial $f \in \mathrm{Int}^{\{r\}}(E, V)$ with*

$$\begin{cases} f(\alpha) \equiv 0 \pmod{\widehat{\mathfrak{m}}} \\ \Phi f(\alpha, \alpha) \equiv 0 \pmod{\widehat{\mathfrak{m}}} \\ \ldots \\ \Phi^r f(\alpha, \ldots, \alpha, \alpha) \equiv 0 \pmod{\widehat{\mathfrak{m}}} \\ \Phi^r f(\alpha, \ldots, \alpha, \beta) \equiv 1 \pmod{\widehat{\mathfrak{m}}}. \end{cases}$$

3.3. $\mathrm{Int}^{\{r\}}(E, V)$ **is not finitely generated.** Considering integer-valued polynomials as functions on $\widehat{E}$, we may view $\mathrm{Int}^{\{r\}}(E, V)$ as contained in $\mathrm{Int}^{\{r\}}(\widehat{E}, \widehat{V})$. We define ideals in $\mathrm{Int}^{\{r\}}(E, V)$ according to the following lemma.

Lemma 3.9. *Let $\alpha \in \widehat{E}$, $s \leq r$, and $\mathfrak{a}$ be an ideal of the completion $\widehat{V}$. Then*

$$\mathfrak{I}_{(\alpha,s,\mathfrak{a})} = \{f \in \mathrm{Int}^{\{r\}}(E, V) \mid f(\alpha) \in \mathfrak{a},\ \Phi f(\alpha, \alpha) \in \mathfrak{a},\ \ldots,\ \Phi^s f(\alpha, \ldots, \alpha) \in \mathfrak{a}\}$$

is an ideal of $\mathrm{Int}^{\{r\}}(E, V)$.

Proof. Since $\Phi^k(f + g) = \Phi^k f + \Phi^k g$, the condition $f, g \in \mathfrak{I}_{(\alpha,s,\mathfrak{a})}$ implies that $(f + g) \in \mathfrak{I}_{(\alpha,s,\mathfrak{a})}$.

Similarly, it follows from the product formula (1.1.3) that

$$\begin{aligned} \Phi^k(fg)(\alpha, \ldots, \alpha) &= f(\alpha)\Phi^k g(\alpha, \ldots, \alpha) \\ &+ \Phi f(\alpha, \alpha)\Phi^{k-1} g(\alpha, \ldots, \alpha) \\ &+ \ldots \\ &+ \Phi^k f(\alpha, \ldots, \alpha) g(\alpha). \end{aligned}$$

Thus, for $f \in \mathfrak{I}_{(\alpha,s,\mathfrak{a})}$ and $g \in \mathrm{Int}^{\{r\}}(E,V)$, we have $\Phi^k(fg)(\alpha,\ldots,\alpha) \in \mathfrak{a}$ for $k \le s$, that is, $fg \in \mathfrak{I}_{(\alpha,s,\mathfrak{a})}$. □

We are now ready to prove the following theorem, which states that rings of the form $\mathrm{Int}^{\{r\}}(E,V)$ are never Noetherian for precompact sets E:

Theorem 3.10. *Let V be a valuation domain with quotient field K and let E be a precompact subset of K. Then $\mathrm{Int}^{\{r\}}(E,V)$ is not Noetherian. If E is infinite and α is a cluster point of E in $\widehat{E}$, then the ideal $\mathfrak{I}_{(\alpha,r,\widehat{\mathfrak{m}})}$ of $\mathrm{Int}^{\{r\}}(E,V)$ is not finitely generated.*

Proof. If E is finite, then, by Proposition 3.14 (to be proved in the next subsection), the ring $\mathrm{Int}^{\{r\}}(E,V)$ does not even satisfy the ascending chain condition on principal ideals and thus is not Noetherian.

Now suppose E is infinite, and by way of contradiction, suppose that $\mathfrak{I}_{(\alpha,r,\widehat{\mathfrak{m}})}$ is finitely generated and let $(h_1,\ldots,h_m)$ be a finite system of generators. By continuity, there exists a neighborhood U of α such that, for each $k \le r$, each $x_0,x_1,\ldots,x_k \in U$, and each of these generators h_i, we have

$$\Phi^k h_i(x_0,\ldots,x_k) \in \widehat{\mathfrak{m}}.$$

By the sum and product formulas 1.1.2 and 1.1.3, the same holds for a linear combination (with coefficients in $\mathrm{Int}^{\{r\}}(E,V)$) of these generators:

$$\forall f \in \mathfrak{I}_{(\alpha,r,\widehat{\mathfrak{m}})},\ \forall k \le r,\ \forall x_0,x_1,\ldots,x_k \in U,\ \Phi^k f(x_0,\ldots,x_k) \in \widehat{\mathfrak{m}}.$$

We thus reach a contradiction with Proposition 3.8 which says there is $f \in \mathfrak{I}_{(\alpha,r,\widehat{\mathfrak{m}})}$, and $\beta \in U$ with $\Phi^r f(\alpha,\ldots,\alpha,\beta) \in \widehat{\mathfrak{m}}$. □

We immediately derive the following.

Corollary 3.11. *Let D be the ring of integers of a number field and E any fractional subset of D. Then $\mathrm{Int}^{\{r\}}(E,D)$ is not Noetherian or a finitely generated D-algebra.*

If the valuation domain V is Noetherian (that is, if the valuation is discrete), it follows immediately from Theorem 3.10 that $\mathrm{Int}^{\{r\}}(E,V)$ is not a finitely generated algebra. In fact, this remains true even if V is not Noetherian.

Proposition 3.12. *Let V be a valuation domain with quotient field K and let E be a precompact subset of K. Then $\mathrm{Int}^{\{r\}}(E,V)$ is not a finitely generated V-algebra.*

Proof. Assume, by way of contradiction, that $\mathrm{Int}^{\{r\}}(E,V) = V[h_1,\ldots,h_m]$. We consider two cases.

•E is finite. Let α be the infimum of the valuations of the coefficients of the generators h_i. Let $f = \sum \beta_j X^j$ be in $V[h_1,\ldots,h_m]$. Since, for each i, the constant term $h_i(0)$ of h_i is in V, it follows that $v(\beta_d) \ge d\alpha$ for each $d > 0$. On the other hand, for nonzero $x \in K$, the polynomial $g_x = x\left(\prod_{a\in E}(X-a)\right)^{r+1}$ belongs to $\mathrm{Int}^{\{r\}}(E,V)$. Indeed, it follows from the product formula 1.1.3 that g_x and its divided differences up to order r are null on E. Now for each x, the polynomial g_x has the same degree d. We thus reach a contradiction as the the valuation of the leading coefficient of g_x is arbitrary.

•E is infinite. Let α be a cluster point of $\widehat{E}$. By continuity of the polynomials h_i, there exists a neighborhood U of α such that for $\beta \in U$ and $j \leq r$,

$$\Phi^j h_i(\alpha, ...\alpha, \beta) \equiv \Phi^j h_i(\alpha, ...\alpha, \alpha) \mod \widehat{\mathfrak{m}}.$$

From the sum and product formulas 1.1.2 and 1.1.3, the same holds for every $f \in \mathrm{Int}^{\{r\}}(E, V)$. As in Theorem 3.10, we obtain a contradiction with Proposition 3.8. □

3.4. **Ascending chain condition on principal ideals.** We have just seen that $\mathrm{Int}^{\{r\}}(E, D)$ is frequently not Noetherian, and $\mathrm{Int}_x(E, D)$ is also sometimes not Noetherian (and each is certainly not if, for instance, D itself is not Noetherian). However, assuming E is infinite in the case of $\mathrm{Int}^{\{r\}}(E, D)$, we prove that both types of rings satisfy the ascending chain condition on principal ideals (for short ACCP) when this is the case for D. Thus, in particular, $\mathrm{Int}^{\{r\}}(D)$ and $\mathrm{Int}_x(D)$ satisfy ACCP. In fact, the same holds for a large class of rings, including $\mathrm{Int}^{\{r\}}(E, D)$, $\mathrm{Int}_x(E, D)$, $\mathrm{Int}^{(r)}(E, D)$ and $\mathrm{Int}^{[r]}(D)$.

Proposition 3.13. *Assume E to be an infinite subset of K and let R be a ring such that $D \subseteq R \subseteq \mathrm{Int}(E, D)$. Then R satisfies ACCP if and only if D satisfies ACCP.*

Proof. The proof is essentially the same as for integer-valued polynomials [7, proposition VI.2.9]. For sake of completeness, we show the condition that D satisfies ACCP to be sufficient. Let $\{g_n R\}$ be an ascending chain of principal ideals. As g_{n+1} divides g_n in R, it does so in $K[X]$. Thus the degree of the polynomials g_n becomes eventually constant: there is N such that, for $n \geq N$, $g_n = a_n g_{n+1}$, with $a_n \in K$. In fact, $a_n \in D$, as $a_n \in R$. As E is infinite, there is $a \in E$ such that $g_0(a) \neq 0$, hence, $g_n(a) \neq 0$ for all n. For $n \geq N$, one has $g_n(a) = a_n g_{n+1}(a)$. Thus $\{g_n(a)\}$ becomes an ascending chain of principal ideals in D and if D satisfies ACCP, a_n is eventually a unit in D. As $D \subseteq R$, a_n is thus a unit in R, and hence, $g_n R = g_{n+1} R$. □

Note that in the case of $\mathrm{Int}_x(E, D)$, it follows from property 1.3.1 that we can replace E by the set of classes met by E modulo xD. Thus, even if E is finite, $\mathrm{Int}_x(E, D)$ satisfies ACCP. On the contrary, if the subset E is finite, we show that $\mathrm{Int}^{\{r\}}(E, D)$ never satisfies ACCP (and thus, a fortiori, is not Noetherian). In fact, this is again true for a large class of domains.

Proposition 3.14. *Assume E to be a finite subset of K and let R be a domain such that $R \cap K = D$ and $\mathrm{Int}^{\{r\}}(E, D) \subseteq R$ for some r. Then R does not satisfy ACCP.*

Note that we impose no restriction on the quotient field of R (which may be any extension of the field $K(X)$ of rational fractions with coefficients in K).

Proof. As in the proof of Proposition 3.12, for nonzero $x \in D$ and for each $n \geq 0$, the polynomial $g_n = \frac{1}{x^n}\left(\prod_{a \in E}(X - a)\right)^{r+1}$ belongs to $\mathrm{Int}^{\{r\}}(E, D)$. Assuming, as always, that D is not a field, we can take x to be a nonunit in D, and thus, since $R \cap K = D$, a nonunit in R. It follows that $\{g_n R\}$ is a strictly ascending chain of principal ideals in R. □

3.5. **The rings** $\mathrm{Int}_x^{\{r\}}(E,D)$**.** Finally, we note that the rings $\mathrm{Int}_x(E,D)$ and $\mathrm{Int}^{\{r\}}(E,D)$ that we have considered in this article have a common generalization $\mathrm{Int}_x^{\{r\}}(E,D)$, defined by

$$\mathrm{Int}_x^{\{r\}}(E,D) = \{f \in K[X] \mid \forall k \le r,\ \forall a \in E^{k+1},\ \Phi^k f(xE^{k+1}+a) \subseteq D\}. \tag{18}$$

In the language of p-adic analysis, such rings arise when constructing orthonormal bases for the $\mathbb{Q}_p$-Banach space of functions on $E \subset \mathbb{Q}_p$ that are r-times continuously differentiable and whose k-th divided differences, for all $k \le r$, are locally analytic of order α (see [5]).

The arguments of this article apply equally well to these combined rings. Namely, we have:

Theorem 3.15. *Let D be a Dedekind domain with quotient field K and E be a locally precompact fractional subset of D. Let $x \neq 0$ be an element of K and r any nonegative integer. Then E possesses a periodic r-removed v-ordering of order $\alpha = v(x)$. The ring $\mathrm{Int}_x^{\{r\}}(E,D)$ is finitely generated over D and is hence Noetherian.*

The above results may be proven as in Section 3.1. If we have a valuation domain V which is not necessarily discrete, then as before the finite generation of $\mathrm{Int}_x^{\{r\}}(E,V)$ depends on the set E and the value group of V. Either way, Proposition 3.13 implies that $\mathrm{Int}_x^{\{r\}}(E,V)$ satisfies ACCP for all values of x in the quotient field K of V and all nonegative integers r.

References

[1] Y. Amice, *Interpolation p-adique*, Bull. Soc. Math. France **92** (1964), 117–180.

[2] D. Barsky, Fonctions k-lipschitziennes sur un anneau local et polynomes à valeurs entières, *Bull. Soc. Math. France* **101** (1973), 397–411.

[3] M. Bhargava, *P-orderings and polynomial functions on arbitrary subsets of Dedekind rings*, J. reine angew. Math. **490** (1997), 101–127.

[4] M. Bhargava and K.S. Kedlaya, Continuous functions on compact subsets of local fields, *Acta Arith.* **91** (1999), 191–198.

[5] M. Bhargava, Analysis on local fields and rings of integer-valued polynomials, preprint.

[6] P.-J. Cahen, *Polynômes et dérivées à valeurs entières*, Ann. Sci. Univ. Clermont, Sér. Math **10**(1975), 25-43.

[7] P.-J. Cahen and J.-L. Chabert, *Integer-Valued Polynomials*, Amer. Math. Soc. Surveys and Monographs, **48**, Providence, 1997.

[8] ———, What's new about Integer-Valued Polynomials on a Subset? in *Non Noetherian Commutative Ring Theory*, Kluwer Academic Press (2000), 75–96.

[9] ———, On the ultrametric Stone-Weierstrass theorem and Mahler's expansion. *Journal de théorie des Nombres de Bordeaux* **14** (2002), 1–15.

[10] P.-J. Cahen, J.-L. Chabert and K. Alan Loper, High dimension Prüfer domains of integral-valued polynomials. *Journal Korean Math. Soc* **38** (2001), 915–935.

[11] L. Carlitz, A note on integral-valued polynomials, Indagationes Math., Ser. A **62** (1959), 294-299.

[12] A. Cauchy, Sur les fonctions interpolaires. *C.R. Acad.Sci. Paris* **11** (1840), 775–789.

[13] J.-L. Chabert, Polynômes à valeurs entières ainsi que leurs dérivées, Ann. Sci. Univ. Clermont, Sér. Math **18**(1979), 47-64.

[14] ———, Dérivées et différences divisées à valeurs entières, Acta Arithmetica **63**(1993), 143-156.

[15] Y. Haouat and F. Grazzini, Polynômes de Barsky, Ann. Sci. Univ. Clermont, Sé. Math. **18**,(1979), 65–81.

[16] J. Yeramian, *Anneaux de Bhargava*, Comm. in Algebra, vol.8 (2002)

Princeton University
E-mail address: bhargava@Math.Princeton.edu

Université Paul Cézanne, Aix-Marseille III, CNRS UMR 6632
E-mail address: paul-jean.cahen@univ-cezanne.fr

Université Paul Cézanne, Aix-Marseille III, CNRS UMR 6632
E-mail address: julie.yeramian@univ-cezanne.fr