ANSWER TO A QUESTION ON THE PRINCIPAL IDEAL THEOREM

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Abstract. We give an example of a finitely generated prime ideal $\mathfrak{p}$ in a domain D with the ascending chain condition on principal ideals, such that $\mathfrak{p}$ is minimal over a (nonzero) principal ideal and satisfies $\bigcap_{n=1}^{\infty} \mathfrak{p}^n = 0$ but such that $\mathrm{ht}\mathfrak{p} > 1$.

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INTRODUCTION

Let D be an integral domain (with 1). It is known that if $\mathfrak{p}$ is a finitely generated prime ideal of D such that $\mathrm{ht}\mathfrak{p} = 1$, then $\bigcap_{n=1}^{\infty} \mathfrak{p}^n = 0$ (indeed this follows from the Krull intersection theorem since the localization $D_{\mathfrak{p}}$ of D with respect to $\mathfrak{p}$ is then a Noetherian local domain). D.D. Anderson et al. ask in [1] whether conversely, if $\mathfrak{p}$ is a finitely generated prime ideal of D which satisfies $\bigcap_{n=1}^{\infty} \mathfrak{p}^n = 0$ and moreover is minimal over a (nonzero) principal ideal, then does $\mathrm{ht}\mathfrak{p} = 1$. We show in this paper that the answer turns out to be negative. Of course, removing the hypothesis that $\mathfrak{p}$ is minimal over a principal ideal, the Krull intersection theorem again implies that any prime ideal $\mathfrak{p}$ of a Noetherian domain such that $\mathrm{ht}\mathfrak{p} > 1$ would give a counterexample. As it is, we want an example in a necessarily non-Noetherian domain that does not satisfy the conclusion of the principal ideal theorem. According to V.Barucci et al. [3, proposition 3.4] a Mori domain (i.e. a domain with the ascending chain condition on divisorial ideals) also satisfies the principal ideal theorem for the prime ideals of finite type. However, we produce a counterexample in a domain D which is "not too far from being Noetherian", in the sense that it satisfies the ascending chain condition on principal ideals (for short D is said to have ACCP).

In proposition 1 we show how to produce counterexamples via a pullback construction which allows "gluing" two maximal ideals of a domain R, each one satisfying some of the requirements. We then show how to obtain such a domain which in addition has ACCP, by considering the ring R_H of integer-valued polynomials on some subset H of the ring of p-adic integers (lemma 2 and proposition 3). For the sake of completeness we then prove in lemma 4 that the ascending chain condition on principal ideals is preserved under the pullback construction.

Terminology is standard as in [9]; we use "$\subset$" to denote proper containment.

Proposition 1 Let R be a domain with two maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ such that

(i) $\mathfrak{m}$ and $\mathfrak{n}$ are finitely generated;

(ii) $\mathrm{ht}\mathfrak{n} = 1$;

(iii) $\mathrm{ht}\mathfrak{m} > 1$ and $\mathfrak{m}$ is minimal over a principal ideal Rx, where $x \in \mathfrak{n}$;

(iv) $R/\mathfrak{m}$ and $R/\mathfrak{n}$ are finite extensions of a field k.

Let $\mathfrak{p} = \mathfrak{m} \cap \mathfrak{n}$ and D be the pullback of the following diagram,

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & k \arrow[d] \\
R \arrow[r] & R/\mathfrak{p}
\end{tikzcd}$$

(thus D is the inverse image of k under the natural map $R \longrightarrow R/\mathfrak{p} \simeq R/\mathfrak{m} \times R/\mathfrak{n}$). Then $\mathfrak{p}$ is a finitely generated maximal ideal of D, $\bigcap_{n=1}^{\infty} \mathfrak{p}^n = 0$, $x \in D$, $\mathfrak{p}$ is minimal over Dx and $\mathrm{ht}\mathfrak{p} > 1$.

Proof. The rings R and D share the ideal $\mathfrak{p}$ [4]. Clearly $\mathfrak{p}$ is a maximal ideal in D (since $D/\mathfrak{p} \simeq k$) and it is a finitely generated ideal in R, since $\mathfrak{p} = \mathfrak{m} \cap \mathfrak{n} = \mathfrak{m}\mathfrak{n}$. Moreover (iv) says that $R/\mathfrak{p}$ is finitely generated as a module over $D/\mathfrak{p} \simeq k$ and therefore R is a finitely generated D-module [4, lemme 1], whence $\mathfrak{p}$ is also a finitely generated ideal in D. Obviously $\bigcap_{n=1}^{\infty} \mathfrak{p}^n = 0$, since $\mathrm{ht}\mathfrak{n} = 1$ and thus $\bigcap_{n=1}^{\infty} \mathfrak{n}^n = 0$. Clearly $x \in \mathfrak{p} = \mathfrak{m} \cap \mathfrak{n}$. Now $\mathfrak{p}$ is minimal over Dx, because if not, there would be a prime $\mathfrak{q}$ in D, containing x, and a chain $(0) \subset \mathfrak{q} \subset \mathfrak{p}$; this chain would lift in R [4, proposition 4], necessarily as $(0) \subset \mathfrak{q}' \subset \mathfrak{m}$, since $\mathrm{ht}\mathfrak{n} = 1$, but then $\mathfrak{m}$ would not be minimal over Rx. Finally, $\mathrm{ht}\mathfrak{p} > 1$, since $\mathrm{ht}\mathfrak{m} > 1$ [4, proposition 5]. $\diamond$

It is easy to give an example of a domain R with two maximal ideals satisfying all the hypotheses of proposition 1, for instance by considering a semi-local Prüfer domain which is the intersection of two discrete incomparable valuation domains, one of rank 1, the other of rank 2. Our next step is to produce an example which also has ACCP.

We denote by Int() the ring of *integer-valued polynomials*. We let p be a prime number, $\mathfrak{O}_p$ be the ring of p-adic integers (i.e. the completion of with respect to the p-adic topology) and ${}_p$ be its quotient ring. We recall that integer-valued polynomials are continuous functions from $\mathfrak{O}_p$ to $\mathfrak{O}_p$ [7] [8]; thus if H is a subset of $\mathfrak{O}_p$, the ring $R_H = \{f \in {}_p[X] \mid f(H) \subseteq \mathfrak{O}_p\}$ contains Int(). We note also that R_H clearly contains the localization ${}_p$ of with respect to p.

Lemma **2** Let H be a subset of $\mathfrak{O}_p$ and $R_H = \{f \in {}_p[X] \mid f(H) \subseteq \mathfrak{O}_p\}$; if the cardinality of H is infinite, then R_H has ACCP.

Proof. This is [2, theorem 7.5]: indeed R_H is such that ${}_p[X] \subseteq R_H \subseteq {}_p + X[X]$, and ${}_p$ has ACCP. Hence we are left to prove that, for any $n \geq 0$, there exists a non-zero element $r_n \in {}_p$ such that $r_n f \in {}_p[X]$ for all $f \in R_H$ with $\deg f \leq n$. Since H is infinite, there are distinct elements $a_0, a_1, \ldots, a_n$ in H. If $f \in R_H$, then $f(a_i) \in \mathfrak{O}_p$, for $0 \leq i \leq n$. If $\deg f = n$, we may consider these $n+1$ conditions as a system of $n+1$ linear equations, whose unknowns are the coefficients of f. Its determinant is the Vandermonde determinant $s_n = \prod_{i<j}(a_j - a_i)$; hence, for any coefficient α of f, $s_n\alpha \in \mathfrak{O}_p$. Denoting by v the p-adic valuation, we thus have $v(\alpha) + v(s_n) \geq 0$. Therefore, if $r_n \in {}_p$ is such that $v(r_n) \geq v(s_n)$, we get $r_n\alpha \in \mathfrak{O}_p \cap = {}_p$. $\diamond$

Our next proposition shows how to meet the requirements of proposition 1. We note first that, since R_H contains ${}_p$, its non-zero prime ideals are of the two following types:

- (i) *the primes above 0*: they are in one-one correspondence with the irreducible polynomials with coefficients in ; if f is such a polynomial, we write $\langle f \rangle$ for the corresponding prime ideal, i.e. $\langle f \rangle = f[X] \cap R_H$;
- (ii) *the primes above p*: they are in one-one correspondence with the elements of the topological closure of H in ${}_p$; to any element α there corresponds the prime $\mathfrak{M}_\alpha = \{f \in R_H \mid f(\alpha) \in p\mathfrak{O}_p\}$ (the proof is as in [5, lemma 3.2]). All these primes are maximal.

Proposition **3** Let H be a subset of $\mathfrak{O}_p$, $R_H = \{f \in {}_p[X] \mid f(H) \subseteq \mathfrak{O}_p\}$, $\alpha \in H$ and $\mathfrak{M}_\alpha = \{f \in R_H \mid f(\alpha) \in p\mathfrak{O}_p\}$. Then

- (i) if α is isolated in H (for the p-adic topology), $\mathfrak{M}_\alpha$ is a finitely generated maximal ideal of the domain R_H;
- (ii) if α is transcendental over , $\mathrm{ht}\mathfrak{M}_\alpha = 1$;
- (iii) if α is algebraic over , $\mathrm{ht}\mathfrak{M}_\alpha = 2$;
- (iv) for any maximal ideal $\mathfrak{M}$ of R_H, $\mathfrak{M}_\alpha$ is minimal over a principal ideal xR_H, where $x \in \mathfrak{M}$;
- (v) for any $\alpha \in H$, $R_H/\mathfrak{M}_\alpha \simeq /p$.

Proof. In analogy with the Stone-Weierstrass theorem, the ring Int() is dense in the ring of continuous functions from $\mathfrak{O}_p$ to $\mathfrak{O}_p$ [8]. Hence, if α is isolated in H, there is an element $f \in \mathrm{Int}() \subseteq R_H$ such that $f(\alpha) \in p\mathfrak{O}_p$ and $f(x) \equiv 1 \pmod{p}$ for all x in H distinct from α. We can conclude that $\mathfrak{M}_\alpha = (f, p)$ and this proves (i): indeed if $g \in \mathfrak{M}_\alpha$ then $\forall x \in H, (g - gf)(x) \in p\mathfrak{O}_p$, thus $g = gf + ph$, where h is such that $h(H) \subseteq \mathfrak{O}_p$, i.e. $h \in R_H$. If α is transcendental over , then $\mathfrak{M}_\alpha$ does not contain any prime above 0, thus $\mathrm{ht}\mathfrak{M}_\alpha = 1$, proving (ii). On the contrary, if α is algebraic, it is the root of an irreducible polynomial f, then $\langle f \rangle \subset \mathfrak{M}_\alpha$ and thus $\mathrm{ht}\mathfrak{M}_\alpha = 2$, proving (iii). Since R_H is an overring of Int(), it is a Prüfer domain [6]. Now, the spectrum of a Prüfer domain R being a tree, it is clear that for any pair of maximal ideals $\mathfrak{m}$ and

$\mathfrak{n}$ of R, $\mathfrak{m}$ is minimal over a principal ideal Rx, where $x \in \mathfrak{n}$; this proves (iv). Lastly (v) is wellknown and anyway obvious. $\diamond$

Taking for instance H to be the union of $p\mathfrak{O}_p$ with two elements α and β of $\mathfrak{O}_p$ in the class of 1 (mod p), α algebraic over and β transcendental, we thus meet all the requirements of proposition 1, with $\mathfrak{m} = \mathfrak{M}_\alpha$ and $\mathfrak{n} = \mathfrak{M}_\beta$ (since α and β are isolated). Moreover, R_H has ACCP, since H is infinite [lemma 2]. Letting $\mathfrak{p} = \mathfrak{M}_\alpha \cap \mathfrak{M}_\beta$ and D be the pullback as in proposition 1, the prime ideal $\mathfrak{p}$ of D is then our counterexample. Moreover D itself has ACCP according to our next and last lemma. Note that $R_H/\mathfrak{p}$ is a Noetherian $/p$-module and thus *a fortiori* satisfies the ascending chain condition on *monogenic* submodules (i.e. submodules generated by a single element); for the sake of brevity, if S is a ring, we say that an S-module M has ACCP if it satisfies the ascending chain condition on monogenic submodules.

Lemma **4** Let R be an ACCP domain, I an ideal of R, S a subring of the quotient R/I and D the pullback of the following diagram.

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & S \arrow[d] \\
R \arrow[r] & R/I
\end{tikzcd}$$

If R is an ACCP domain and R/I is an ACCP S-module, then D is an ACCP domain.

Proof. Consider an ascending chain $Dx_1 \subseteq \ldots \subseteq Dx_n \subseteq Dx_{n+1} \subseteq \ldots$ of principal ideals. Since R has ACCP, there is an integer m such that, for $n \geq m$, $Rx_m = Rx_n$, thus $D \subseteq D(x_n/x_m) \subseteq R$. But since R/I is an ACCP S-module (and since $S \simeq D/I$), there is an integer $m' \geq m$ such that, for $n \geq m'$, $D(x_n/x_m) = D(x_{m'}/x_m)$ or equivalently $Dx_n = Dx_{m'}$. $\diamond$

References

[1] D.D.Anderson, J. Matijevic and W.Nichols. The Krull Intersection Theorem II. *Pacific J. Math.* **66** (1976), 15-22.

[2] D.D.Anderson, D.F.Anderson and M.Zafrullah. Rings between $D[X]$ and $K[X]$. *Houston J. Math.* **17** (1991), 109-129.

[3] V.Barucci, D.F.Anderson and D.E.Dobbs. Coherent Mori domains and the principal ideal theorem. *Comm. in Algebra* **15 (6)** (1987), 1119-1156.

[4] P.J.Cahen. Couples d'anneaux partageant un idéal. *Archiv der Math.* **51** (1988), 505-514.

[5] P.J.Cahen. Integer-valued polynomials on a subset. *Proc. Amer. Math. Soc.* **117** (1993), 919-929.

[6] J.L.Chabert. Un anneau de Prüfer. *J. Algebra* **107** (1987), 1-16.

[7] J.L.Chabert. Les idéaux premiers de l'anneau des polynômes à valeurs entières. *J. Reine Angew. Math.* **293/294** (1977), 275-283.

[8] K.Mahler. An interpolation series for continuous functions of a p-adic variable. *J. Reine Angew. Math.* **199** (1958), 23-34.

[9] R.Gilmer. *Multiplicative ideal theory.* Marcel Dekker, New York, 1974.