

Polynomial Closure

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Abstract. Let D be a domain with quotient field K . The polynomial closure of a subset E of K is the largest subset F of K such that each polynomial (with coefficients in K), which maps E into D , maps also F into D . In this paper we show that the closure of a fractional ideal is a fractional ideal, that divisorial ideals are closed and that conversely closed ideals are divisorial for a Krull domain. If D is a Zariski ring, the polynomial closure of a subset is shown to contain its topological closure; the two closures are the same if D is a one-dimensional Noetherian local domain, with finite residue field, which is analytically irreducible. A subset of D is said to be polynomially dense in D if its polynomial closure is D itself. The characterization of such subsets is applied to determine the ring R_α formed by the values $f(\alpha)$ of the integer-valued polynomials on a Dedekind domain R (at some element α of an extension of R). It is also applied to generalize a characterization of the Noetherian domains D such that the ring $\text{Int}(D)$ of integer-valued polynomials on D is contained in the ring $\text{Int}(D')$ of integer-valued polynomials on the integral closure D' of D .

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Introduction

A few years ago R. Gilmer [13] characterized the subsets S of the ring of integers such that every polynomial taking integer values on S takes in fact integer values on every integer. This paper was immediately followed by comments by D. McQuillan [17] and ourselves [3]. Letting D be a domain, E a subset of its quotient field K , and

$$\text{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\},$$

we discussed the subsets E of D such that

$$\text{Int}(E, D) = \text{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}.$$

We thus recovered easily Gilmer's result using topological arguments. More generally, McQuillan proposed to discuss when $\text{Int}(E_1, D) = \text{Int}(E_2, D)$ for two subsets E_1 and E_2 . He observed that, for a given subset E , there is a largest subset, containing E , that defines the same ring of *integer-valued polynomials*. He thus defined and studied a closure operation, from which he recovered Gilmer's result (but with somewhat technical devices applying only to Dedekind domains with finite residue fields).

In this paper we study this *polynomial closure* in a general setting. Throughout, we let D be a domain (which is not a field), with quotient field K , and we denote by D' its integral closure.

We let the polynomial D -closure of a subset E of K be the largest subset F of K such that $\text{Int}(E, D) = \text{Int}(F, D)$. After some generalities (particularly on union and intersection), we discuss in a first section when the D -closure of E is K or equivalently $\text{Int}(E, D)$ is *trivial* (that is, does not contain any non constant polynomial). We show that, if D' is a fractional subset of D , then $\text{Int}(E, D)$ is *not* trivial if and only if E is a fractional subset of D (that is, is contained in a finitely generated D -module). In the next section we show that the closure of a fractional ideal is a fractional ideal and also that the divisorial ideals are always closed (generalizing McQuillan's result that each ideal of a Dedekind domain with finite residue fields is closed [17]). However we give examples of divisorial ideals which are not closed (even if D is Noetherian). We then look at localization properties. In particular, we show that integer-valued polynomials on a Krull domain have a somewhat good behavior under localization at the height-one primes (similarly to integer-valued polynomials on a Noetherian domain under localization at every prime). We conclude that the D -closure of a subset E of a Krull domain (resp., a Noetherian domain) is the intersection of its *local* $D_{\mathfrak{p}}$ -closures, where \mathfrak{p} runs among the height-one primes (resp., the maximal ideals) of D . It results that for a Krull domain, a fractional ideal is polynomially closed if and only if it is divisorial.

In the fourth and next section we then use the topological tools we had previously introduced. If D is a (Noetherian) Zariski domain (in particular if D

is a local ring, with maximal ideal \mathfrak{m} and if we consider the \mathfrak{m} -adic topology), we show that the polynomial closure of a fractional subset contains its topological closure. The two closures are the same if D is a one-dimensional Noetherian local domain, with finite residue field, which is analytically irreducible (hence in particular if D is a discrete rank-one valuation domain). We thus recover and generalize McQuillan's characterization of the closure of a subset, when D is a Dedekind domain with finite residue fields.

A subset of D is said to be *polynomially dense* in D if its polynomial closure is D itself. Such subsets were studied in [3] (under the name of *parties pleines de D*) and polynomially dense subrings R of D had previously been characterized, originally in [8], then in [11] and [12] (as *sous-anneaux substituables à D*), in the cases where R or D (or both) are Dedekind domains. These characterizations allow us, in the next section, to interpret and generalize another result of D. McQuillan [18]. If R is a domain, we let R_α be the ring formed by the values $f(\alpha)$ of the integer-valued polynomials on R at some element α of an extension D of R . If R is a Dedekind domain, we show that R_α is the smallest subring of D , containing $R[\alpha]$, in which R is polynomially dense. If R is a Dedekind domain with finite residue fields and α is algebraic over R , then R_α contains the integral closure S of R in $K(\alpha)$ and we recover McQuillan's result according which R_α is the smallest overring of S , containing α , in which the primes of R split completely.

We lastly discuss integral closure. It is known that $D'[X]$ is the integral closure of $D[X]$. However, $\text{Int}(D)$ need not even be contained in $\text{Int}(D')$. R. Gilmer et al. have characterized the *one-dimensional* Noetherian domains for which this inclusion holds [14]. We generalize this characterization, whatever the dimension of D , showing that equivalently D is then a polynomially dense subset of D' . In fact, if D is Noetherian, the integral closure of $\text{Int}(D)$ is the ring $\text{Int}(D, D')$ and if D is a one-dimensional Noetherian domain with finite residue fields, then $\text{Int}(E, D')$ is the integral closure of $\text{Int}(E, D)$, for each fractional subset E of D . We conclude with examples showing that, if D is not Noetherian, then $\text{Int}(D, D')$ is not necessarily integral over $\text{Int}(D)$, and that $\text{Int}(E, D')$ is not necessarily integral over $\text{Int}(E, D)$, even if D is a one-dimensional Noetherian local domain, when its residue field is infinite.

Throughout we let the symbol " \subset " denote proper containment and the symbol " \subseteq " denote large containment.

1 Equivalent subsets and polynomial closure

Definitions 1.1 1. Two subsets E and F of K are said to be *polynomially D -equivalent* (or simply *equivalent* if the context is clear) if $\text{Int}(E, D) = \text{Int}(F, D)$.

2. If E is a subset of K , we denote by \overline{E} the subset

$$\overline{E} = \{x \in K \mid \forall f \in \text{Int}(E, D), f(x) \in D\}$$

and call it the *polynomial D -closure* of E (or simply the *polynomial closure* and even the *closure* of E).

3. If a subset E of K is equal to its polynomial closure, we say that E is *polynomially D -closed* (or simply *(polynomially) closed*).

4. If a subset E of D is D -equivalent to D , we say that E is a *polynomially dense* subset of D .

As noted by D. McQuillan [17], the closure \overline{E} of E is clearly the largest subset of K equivalent to E . Right after introducing this polynomial closure (for Dedekind domains with finite residue fields), McQuillan noted a few easy facts [17, p. 246]. We expand here on them as follows.

Lemma 1.2 1. For each subset E of K , \overline{E} is polynomially closed.

2. If $E \subseteq F$, then $\overline{E} \subseteq \overline{F}$.

3. If (E_i) is a family of subsets of K , then $\overline{\bigcap_i E_i} \subseteq \bigcap_i \overline{E_i}$ and $\bigcup_i \overline{E_i} \subseteq \overline{\bigcup_i E_i}$.

4. For each $a \in K$, $a\overline{E} = \overline{aE}$ and $a + \overline{E} = \overline{a + E}$.

5. Each finite subset E of K is polynomially closed.

Proof. The first two assertions are obvious and the third one can easily be derived. For the next one McQuillan argued that the maps $f(X) \rightarrow f(a + X)$ and $f(X) \rightarrow f(aX)$ define automorphisms of $K[X]$ for each $a \in K$ (the latter for $a \neq 0$, but if $a = 0$, then anyway $0\overline{E} = \overline{0E} = 0$). Lastly, let $E = x_1, \dots, x_n$ be a finite subset of K , $x \notin E$, f the polynomial $f = \prod_{i=1}^n (X - x_i)$ and $\alpha = f(x)$. Since D is not a field, there is $\beta \in K$ such that $\alpha\beta \notin D$. Then $\beta f \in \text{Int}(E, D)$ but $\beta f(x) \notin D$, hence $x \notin \overline{E}$. \diamond

Remarks 1.3 1. As stated by McQuillan [17, p. 246], it results in particular from assertion 3 that the intersection of closed sets is closed.

2. In general $\overline{\bigcap_i E_i}$ is strictly contained in $\bigcap_i \overline{E_i}$. For example both closures of $^+ = \{n \in \mathbb{Z} \mid n \geq 0\}$ and $^- = \{n \in \mathbb{Z} \mid n \leq 0\}$ are equal to \mathbb{Z} (from Gilmer's

characterization of the polynomially dense subsets of [13, Theorem 2]). However there intersection is $\{0\}$ (a finite thus a closed subset).

3. In general also $\bigcup_i \overline{E_i}$ is strictly contained in $\overline{\bigcup_i E_i}$. In particular the finite union of closed sets need not be closed. For example, the ideals of \mathbb{Z} are closed [17, Corollary 1] and [Proposition 2.1 below], whereas the union of two coprime ideals of \mathbb{Z} is polynomially dense in \mathbb{Z} [Corollary 3.14 below]. Hence, as noted by McQuillan [17, Remark 1], the polynomially closed sets do not define a topology.

Now we comment on subsets E of K such that $\text{Int}(E, D)$ is *trivial*, that is, equal to D , in other words E is D -equivalent to K (since one may check easily — or derive from the next proposition — that $\text{Int}(K, D) = D$). Recall that a subset E of K is said to be a *fractional* subset of D if there is a non-zero element d of D such that $dE \subseteq D$. In the case of a Dedekind domain with finite residue fields, D. McQuillan proved that E is D -equivalent to K if and only if E is not fractional [17, Theorem 1]. With a very similar argument we prove here more generally the following.

Proposition 1.4 1. $\text{Int}(E, D)$ contains a polynomial of degree 1 if and only if E is a fractional subset of D .

2. If $\text{Int}(E, D)$ contains a non-constant polynomial then E is a fractional subset of the integral closure D' of D .

Proof. The first assertion is easily derived from the fact that dE is contained in D if and only if the polynomial dX belongs to $\text{Int}(E, D)$. As for the second, let $f = \sum_{i=0}^n a_i X^i$ be a degree n polynomial in $\text{Int}(E, D)$. We may as well assume f to be in $D[X]$ (multiplying by a common denominator of its coefficients). For each $x \in E$, $a_n^{n-1} f(x) \in D$, thus $(a_n x)^n + a_{n-1} (a_n x)^{n-1} + \dots + a_0 (a_n)^{n-1} = b$, where $b \in D$, hence $a_n x \in D'$. \diamond

Corollary 1.5 If D' is a fractional subset of D , then $\text{Int}(E, D)$ contains a non-constant polynomial if and only if E is a fractional subset of D .

This applies in particular if D is integrally closed. It does also apply a fortiori if the complete integral closure D'' of D is a fractional subset of D , a situation which is very common when D is a pullback.

Remark 1.6 If E is a fractional subset of D , then dE is a subset of D and $f(X) \in \text{Int}(dE, D)$ if and only if $f(dX) \in \text{Int}(E, D)$. Thus, for most of the results coming next, where E is supposed to be fractional, we may as well assume it is a subset of D .

It is worth giving examples such that 1) E is *not* a fractional subset of D but $\text{Int}(E, D)$ contains some non-constant polynomials (however no polynomial of degree 1, according to the previous proposition) 2) E is a fractional subset of

D' although $\text{Int}(E, D)$ does not contain any non-constant polynomials. In both cases E is taken to be D' and D' is not a fractional subset of D , hence not a finitely generated D -module. The first example is quite classical, however we describe it completely for the sake of completeness.

Examples 1.7 1. An example of a (dimension 1 and local) Noetherian domain D such that D' is not a finitely generated D -module and $\text{Int}(D', D)$ contains a non-constant polynomial.

Let k be a field with characteristic p and $V = k[[x]]$ the power series ring with coefficients in k . Then V is a rank-one discrete valuation ring and we denote by v the corresponding valuation on its quotient field $k((x))$. It is known that $k((x))$ is a transcendental extension of $k(x)$, so let y be an element which is transcendental. We may assume that $y \in V$ (multiplying if necessary by a power of x). Let $K = k(x, y^p)$ and $L = k(x, y)$, then $K \subset L \subset k((x))$, L is a (purely inseparable) algebraic extension of degree p of K and $L^p \subseteq K$. Let $W = V \cap K$, then W is the ring of the restriction w of the valuation v to K , it is a rank-one discrete valuation domain. Let $D = W[y]$, then D is a Noetherian domain with field of quotient L . We claim its integral closure D' is the intersection $D' = V \cap L$ (that is, the ring of the restriction of v to L). First, it is clear that $V \cap L$ contains W and y , thus contains D . Since $L^p \subseteq K$, then $(V \cap L)^p \subseteq W$ and $V \cap L$ is integral over W . A fortiori $V \cap L$ is integral over D , hence it is contained in D' . Conversely, D' is contained in every valuation overring of D , thus in $V \cap L$. Moreover, the valuation w (with ring W) extends uniquely to L (the ring of the extension being D'). Since $x \in K$, it is easy to check that the ramification index and the residual degree of the extension $W \subset V$ are both 1. A fortiori the ramification index e and the residual degree f of the extension $W \subset D'$ are again 1. Thus $ef = 1$, although L is an extension of degree p of K , and therefore D' is not a finitely generated W -module (e.g. see [1, VI, §8, Theorem 2]). Hence D' is not a finitely generated D -module since $D = W[y]$ is a finitely generated W -module. But lastly, the polynomial X^p belongs to $\text{Int}(D', D)$, since $D'^p \subseteq W \subseteq D$.

2. An example of a domain D such that $\text{Int}(D', D)$ does not contain any non-constant polynomial.

Let k be a field, $B = k[x_1, x_2, \dots]$ the ring in infinitely many indeterminates with coefficients in k (indexed by the positive integers) and D the subring generated by all the elements x_n^r , where $r \geq n$. Clearly D and B have the same quotient field K (since $x_n = x_n^{n+1}/x_n^n$), thus $B = D'$ is the integral closure of D . Now suppose, by way of contradiction, that a non-constant polynomial $f \in K[X]$ is such that $f(B) \subseteq D$. We may assume that the coefficients of f are in D (multiplying them, if need be, by their common denominator). Each of these coefficients is a polynomial in finitely many indeterminates. Let n be larger than the degree d of f and such that x_n is not any of these indeterminates. Write $f(X) = g_d X^d + \dots + g_1 X + g_0$. In particular $f(x_n)$ should belong to D , whereas $f(x_n) = g_d x_n^d + \dots + g_1 x_n + g_0$ is not in D .

2 Polynomial closure of an ideal

For a Dedekind domain with finite residue fields, McQuillan asserted that every ideal is closed [17, Corollary 1]. For every domain, we have in fact easily the following.

Proposition 2.1 *Each divisorial ideal of D is polynomially closed.*

Proof. Since the polynomial X is in $\text{Int}(D)$, it is clear that D is polynomially closed. So is each fractional principal ideal, from assertion 4 of Lemma 1.2 and so is each divisorial from assertion 3 of this lemma, since a divisorial ideal is the intersection of fractional principal ideals. \diamond

Remarks 2.2 1. A subset is a fractional subset of D if and only if it is contained in a (non-zero) principal fractional ideal. Hence it results from the previous proposition that the closure of a fractional subset is a fractional subset.

2. The converse of the previous proposition does not hold. Generalizing McQuillan's result in another direction, we shall see below that every ideal of D is closed when D is a one-dimensional, Noetherian, locally analytically irreducible domain with finite residue fields [Corollary 4.8]. However an ideal of D is not necessarily divisorial in this case. For example, if $D = k[[t^3, t^4, t^5]]$, the ring of series, with coefficients in a finite field k and no term in t nor t^2 (a local analytically irreducible domain with finite residue field), one may check that the ideal $\mathfrak{a} = (t^3, t^4)$ is not divisorial.

Although the converse of Proposition 2.1 does not hold in general, we shall prove in the next section that it does for a Krull domain [Proposition 3.8]. We also prove here that we have the following.

Proposition 2.3 *Let D be a local Noetherian domain with maximal ideal \mathfrak{m} . Then \mathfrak{m} is polynomially closed if and only if it is a divisorial ideal.*

Proof. Assume \mathfrak{m} not to be divisorial, then it is not the conductor of a non-zero element of K in D , that is, it is not an associated prime of the D -module K/D . Necessarily the dimension of D is more than one and $\text{Int}(D)$ is *trivial*, that is, equal to $D[X]$ [5, Corollaire 1, p. 297]. We wish to prove that \mathfrak{m} is not closed hence that $\text{Int}(\mathfrak{m}, D) = D[X]$. Let $f \in \text{Int}(\mathfrak{m}, D)$ and write $f = \alpha_0 + \alpha_1 X + \dots + \alpha_n X^n$. Suppose, by way of contradiction, that some coefficient α_k of f were not in D , its conductor $\mathfrak{a}_k = [D : \alpha_k]$ would then be a proper ideal of D . For all $x \in \mathfrak{m}$, $f(xX) \in D[X]$ (indeed, if $a \in D$, then $xa \in \mathfrak{m}$, thus $f(xa) \in D$ and $f(xX) \in \text{Int}(D)$). Thus in particular $\alpha_k x^k \in D$. Hence \mathfrak{m} would be minimal among the primes containing $\mathfrak{a}_k = [D : \alpha_k]$, which is the annihilator of a non-zero element of the D -module K/D . Since D is Noetherian, \mathfrak{m} would then be an associated prime of K/D , contrary to our assumption. \diamond

Although, in general, some ideals are not closed, we show now that the polynomial closure of an ideal is an ideal but first prove the following, where the sum $E + F$ (resp., the product EF), of two subsets E and F is the set of the elements of the form $x + y$ (resp., xy), where $x \in E, y \in F$.

Lemma 2.4 *Let E and F be two subsets of K , then*

1. $\overline{E} + \overline{F} \subseteq \overline{E + F}$.
2. $\overline{E} \overline{F} \subseteq \overline{EF}$.

Proof. From assertion 4 of Lemma 1.2, if $\alpha \in E$, then $\alpha + \overline{F} = \overline{\alpha + F}$. Thus $E + \overline{F} \subseteq \overline{E + F}$. By symmetry, we then get

$$\overline{E} + \overline{F} \subseteq \overline{E + F} \subseteq \overline{\overline{E + F}} = \overline{E + F}.$$

The second assertion is perfectly similar. \diamond

Remark 2.5 These inclusions are strict in general: 1. The sum of two closed sets is not necessarily closed. For example, let $D = k[[x, y]]$ be the power series ring with coefficients in the field k , in two indeterminates x and y . The principal ideals (x) and (y) are closed [Proposition 2.1] whereas the maximal ideal $(x) + (y)$ is not [Proposition 2.3].

2. The product of two closed sets is not necessarily closed. For example, in \mathbb{Z} , consider the subsets $E = \{2, 3\}$ and $F = \mathbb{Z}$. Then E is closed since it is finite and so is obviously F [Proposition 2.1]. However their product is the union $2\mathbb{Z} \cup 3\mathbb{Z}$, which is polynomially dense in \mathbb{Z} [Corollary 3.14 below].

Proposition 2.6 *The polynomial closure of an additive subgroup (resp., a subring, resp., a fractional ideal) of D is an additive subgroup (resp., a subring, resp., a fractional ideal) of D .*

Proof. It results for example from the previous lemma that, if \mathfrak{a} is an ideal of D , then $\overline{\mathfrak{a}} + \overline{\mathfrak{a}} \subseteq \overline{\mathfrak{a} + \mathfrak{a}} = \overline{\mathfrak{a}}$, and $D\overline{\mathfrak{a}} \subseteq \overline{D\mathfrak{a}} = \overline{\mathfrak{a}}$. \diamond

Remark 2.7 In his paper D. McQuillan also considers *homogeneous sets* that he defines to be the (finite) union of cosets modulo an (integral) ideal of D . He then claims that the closure of an homogeneous set (for a Dedekind domain with finite residue fields) is again homogeneous with respect to the same ideal [17, Lemma 3]. More generally we may note that an homogeneous set with respect to the ideal \mathfrak{a} is a subset E such that $E + \mathfrak{a} \subseteq E$ (we do not insist that E be a finite union of cosets modulo \mathfrak{a} , a fact which is particular to Dedekind domains with *finite* residue fields). We may then take this inclusion as a definition and then allow \mathfrak{a} to be a fractional ideal. It is then immediate to derive from Lemma 2.4 that the closure of E is homogeneous with respect to the closure $\overline{\mathfrak{a}}$ of \mathfrak{a} (in the case of a Dedekind domain, each ideal is closed and \overline{E} is then homogeneous with respect to the same ideal \mathfrak{a}).

3 Localization

Let us first recall a few elementary facts we had previously established [4, Proposition 1.5]: if E is a subset of K and S a multiplicative subset of D , then $S^{-1}\text{Int}(E, D)$ is clearly a subring of $\text{Int}(E, S^{-1}D)$. Under Noetherian hypotheses we showed there was equality. We restate the following for sake of completeness:

Proposition 3.1 *1. Let D be a Noetherian domain, E a fractional subset of D and S a multiplicative subset of D , then*

$$S^{-1}\text{Int}(E, D) = \text{Int}(E, S^{-1}D).$$

2. Let D be a domain, R a Noetherian subring of D and S a multiplicative subset of R , then

$$S^{-1}\text{Int}(R, D) = \text{Int}(R, S^{-1}D) = \text{Int}(S^{-1}R, S^{-1}D).$$

Similarly we prove here that we have a good behavior under localization at the height-one primes of a Krull domain.

Proposition 3.2 *Let D be a Krull domain, \mathfrak{q} an height-one prime and E a fractional subset of D . Then*

$$\text{Int}(E, D)_{\mathfrak{q}} = \text{Int}(E, D_{\mathfrak{q}}).$$

Proof. We wish to prove that $\text{Int}(E, D_{\mathfrak{q}}) \subseteq \text{Int}(E, D)_{\mathfrak{q}}$. Let $f \in \text{Int}(E, D_{\mathfrak{q}})$. There is $d \in D, d \neq 0$, such that $df \in D[X]$. Let \mathcal{W} be the (finite) set of essential valuations w of D such that $w(d) > 0$ and w is not associated with \mathfrak{q} . For each height-one prime \mathfrak{p} distinct from \mathfrak{q} , there is an element of \mathfrak{p} which is not in \mathfrak{q} . Taking a product of such elements, we can thus get $a \in D, a \notin \mathfrak{q}$ such that, for each $w \in \mathcal{W}, w(a) \geq w(d)$. Hence, for each height-one prime \mathfrak{p} , but $\mathfrak{q}, af \in D_{\mathfrak{p}}[X]$. Assuming E to be a subset of D [Remark 1.6], then $af(E)$ is contained in $D_{\mathfrak{p}}$. But $af(E)$ is also contained in $D_{\mathfrak{q}}$, since $a \in D$ and $f \in \text{Int}(E, D_{\mathfrak{q}})$ by hypothesis. Therefore $af \in \text{Int}(E, D)$, hence $f \in \text{Int}(E, D)_{\mathfrak{q}}$, since $a \notin \mathfrak{q}$. \diamond

Remark 3.3 Let us note that in Propositions 3.1 and 3.2, it is essential to assume E to be a fractional subset. Indeed the ring of integers is obviously both a Noetherian and a Krull domain. But, for each nontrivial multiplicative subset S of $\mathbb{Z}, E = S^{-1}$ is not a fractional subset of \mathbb{Z} . Therefore $S^{-1}\text{Int}(E, \mathbb{Z}) = S^{-1}$ [Corollary 1.5], whereas $\text{Int}(E, S^{-1}) = \text{Int}(S^{-1})$, a ring which contains non-constant polynomials.

If E is a subset of K and S a multiplicative subset (resp., \mathfrak{p} a prime ideal) of D , we now consider the polynomial $S^{-1}D$ -closure (resp., the polynomial $D_{\mathfrak{p}}$ -closure) of E , that we denote by $\overline{S^{-1}E}$ (resp., $\overline{E_{\mathfrak{p}}}$). If $E = \mathfrak{a}$ is a fractional ideal of D , there could be some confusion with the $S^{-1}D$ -closure of $S^{-1}\mathfrak{a}$, however next lemma shows there is no harm (there could still be some confusion with the closure of $S^{-1}\mathfrak{a}$ in D , but we do believe the context makes clear what we have in mind).

Lemma 3.4 *Let \mathfrak{a} be a fractional ideal and S be a multiplicative subset of D . Then \mathfrak{a} and $S^{-1}\mathfrak{a}$ are $S^{-1}D$ -equivalent.*

Proof. Let $f(X) \in \text{Int}(\mathfrak{a}, S^{-1}D)$ and $(x/s) \in S^{-1}\mathfrak{a}$. Since $xD \subseteq \mathfrak{a}$, then $f(X) \in \text{Int}(xD, S^{-1}D)$, thus $f(xX) \in \text{Int}(D, S^{-1}D) = \text{Int}(S^{-1}D)$ [5, Corollaire 5, p. 303]. Hence $f(X) \in \text{Int}(xS^{-1}D, S^{-1}D)$ and $f(x/s) \in S^{-1}D$. \diamond

We now relate the polynomial D -closure \overline{E} of E to its *local* $S^{-1}D$ -closures.

Proposition 3.5 *Let E be a subset of K .*

1. *Let S be a multiplicative subset of D and assume that*

$$S^{-1}\text{Int}(E, D) = \text{Int}(E, S^{-1}D),$$

then $\overline{E} \subseteq \overline{S^{-1}E}$.

2. *Let $(S_i)_{i \in I}$ be a complete family of multiplicative subsets of D , that is, such that $D = \bigcap_{i \in I} S_i^{-1}D$, then $\bigcap_{i \in I} \overline{S_i^{-1}E} \subseteq \overline{E}$.*

Proof. 1. Let $x \in \overline{E}$, and $f \in \text{Int}(E, S^{-1}D) = S^{-1}\text{Int}(E, D)$. There is $s \in S$ such that $sf \in \text{Int}(E, D)$, thus $sf(x) \in D$ and $f(x) \in S^{-1}D$. In conclusion $x \in \overline{S^{-1}E}$.

2. Let $x \in \bigcap_{i \in I} \overline{S_i^{-1}E}$ and $f \in \text{Int}(E, D)$. Then, $\forall i \in I$, $f \in \text{Int}(E, S_i^{-1}D)$, thus $f(x) \in S_i^{-1}D$. In conclusion $f(x) \in D = \bigcap_{i \in I} S_i^{-1}D$. \diamond

Using Propositions 3.1 and 3.2, we then derive the following.

Proposition 3.6 1. *If D is Noetherian and \mathcal{M} is the set of its maximal ideals, then $\overline{E} = \bigcap_{\mathfrak{m} \in \mathcal{M}} \overline{E_{\mathfrak{m}}}$.*

2. *If D is a Krull domain and \mathcal{P} is the set of its height-one primes, then $\overline{E} = \bigcap_{\mathfrak{p} \in \mathcal{P}} \overline{E_{\mathfrak{p}}}$.*

In particular, we thus get the following (the Noetherian part of which we had already established [3, Proposition 1.3]).

Corollary 3.7 *Let E be a subset of D .*

1. If D is Noetherian, then E is polynomially dense in D if and only if it is polynomially dense in $D_{\mathfrak{m}}$, for each maximal ideal \mathfrak{m} of D .
2. If D is a Krull domain, then E is polynomially dense in D if and only if it is polynomially dense in $D_{\mathfrak{p}}$, for each height-one prime ideal \mathfrak{p} of D .

For a Krull domain D , we may note that, whatever the fractional ideal \mathfrak{a} (whether it is closed or not closed), $\mathfrak{a}_{\mathfrak{p}}$ is a divisorial ideal of $D_{\mathfrak{p}}$ (a discrete rank-one valuation domain), for each height-one prime ideal of D , hence is a $D_{\mathfrak{p}}$ -closed ideal. From Proposition 3.6, the polynomial closure of \mathfrak{a} is then the intersection $\bar{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in P} \mathfrak{a}_{\mathfrak{p}}$ and this intersection is known to be the v -closure of \mathfrak{a} [1, VII, §1, Proposition 7] (that is, the smallest divisorial ideal containing \mathfrak{a}). We thus get the following (in particular the converse to Proposition 2.1 holds in this case).

Corollary 3.8 *Let D be a Krull domain and \mathfrak{a} be a fractional ideal of D . The polynomial closure of \mathfrak{a} is then the v -closure of \mathfrak{a} . In particular \mathfrak{a} is polynomially D -closed if and only if it is a divisorial ideal of D .*

In general, a D -closed subset E is not $S^{-1}D$ -closed, indeed the $S^{-1}D$ -closure of an ideal \mathfrak{a} contains $S^{-1}\mathfrak{a}$. It may even happen that an ideal \mathfrak{a} is D -closed whereas $S^{-1}\mathfrak{a}$ is not $S^{-1}D$ -closed, according to the next example (which stems from a classical pullback construction).

Example 3.9 Let $B = [[x, y]]$, the power series ring over the field of rationals. This is a Noetherian Krull domain, its maximal ideal $\mathfrak{m} = (x, y)$ is not divisorial hence it is not B -closed from the previous corollary (or Proposition 2.3). Let $D = B + \mathfrak{m}$. D is a subring of B and \mathfrak{m} is a prime ideal of D . It is easy to check that $D_{\mathfrak{m}} = B$, indeed, on the one hand, $D_{\mathfrak{m}}$ is clearly contained in $B_{\mathfrak{m}} = B$, on the other (and thus also $B = B + \mathfrak{m}$) is contained in $D_{\mathfrak{m}}$. In particular $\mathfrak{m}D_{\mathfrak{m}} = \mathfrak{m}$ and $\mathfrak{m}D_{\mathfrak{m}}$ is not $D_{\mathfrak{m}}$ -closed. However, for each non-zero integer a , the ideal \mathfrak{m} is contained in the principal ideal Da , in fact $\mathfrak{m} = ma$, indeed a is invertible in B and $\forall m \in \mathfrak{m}, m = (m/a)a$, where $(m/a) \in \mathfrak{m}$. The ideal \mathfrak{m} is then the intersection of principal ideals in D (since in the quotient $D/\mathfrak{m} \simeq \mathbb{Q}$, the ideal (0) is the intersection of principal ideals.). Therefore \mathfrak{m} is a divisorial ideal of D and thus is D -closed.

However D is not Noetherian in the previous example and we ask the following.

Question 3.10 If D is Noetherian and \mathfrak{a} is a D -closed fractional ideal, is $S^{-1}\mathfrak{a}$ a $S^{-1}D$ -closed ideal ?

Conversely, we may derive immediately the following from Proposition 3.6.

Corollary 3.11 *Let \mathfrak{a} be a fractional ideal of a Noetherian domain D . If $\mathfrak{a}_{\mathfrak{m}}$ is polynomially $D_{\mathfrak{m}}$ -closed, for each maximal ideal \mathfrak{m} of D , then \mathfrak{a} is polynomially D -closed.*

For a Dedekind domain, with finite residue fields, McQuillan proved that the closure of the union $\mathfrak{a} \cup \mathfrak{b}$ of two fractional ideals is their *greatest common divisor* [17, Corollary 2]. More generally we get here the following.

Corollary 3.12 *Let D be a Krull or a Prüfer domain, and $\mathfrak{a}, \mathfrak{b}$ be two fractional ideals of D . Then $\overline{\mathfrak{a} \cup \mathfrak{b}} = \overline{\mathfrak{a} + \mathfrak{b}}$. In particular, if \mathfrak{a} and \mathfrak{b} are finitely generated and D is a Prüfer domain, then $\overline{\mathfrak{a} \cup \mathfrak{b}} = \mathfrak{a} + \mathfrak{b}$.*

Proof. The closure $\overline{\mathfrak{a} \cup \mathfrak{b}}$ of the union is clearly contained in the closure $\overline{\mathfrak{a} + \mathfrak{b}}$ of the sum. Conversely, from Proposition 3.5, $\overline{\mathfrak{a} \cup \mathfrak{b}}$ contains the intersection $\bigcap_{\mathfrak{p} \in \mathcal{P}} (\mathfrak{a} \cup \mathfrak{b})_{\mathfrak{p}}$ of the local closures at each height-one prime (resp., maximal ideal) \mathfrak{p} of D if D is a Krull (resp., a Prüfer) domain. Now, for each such prime, $D_{\mathfrak{p}}$ is a valuation domain, hence one of the ideals $\mathfrak{a}_{\mathfrak{p}}$ or $\mathfrak{b}_{\mathfrak{p}}$ is contained in the other and $\mathfrak{a}_{\mathfrak{p}} + \mathfrak{b}_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}} \cup \mathfrak{b}_{\mathfrak{p}}$. Thus $(\mathfrak{a} \cup \mathfrak{b})_{\mathfrak{p}}$, which contains both closures of \mathfrak{a} and \mathfrak{b} , contains both $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{b}_{\mathfrak{p}}$, hence their sum $\mathfrak{a}_{\mathfrak{p}} + \mathfrak{b}_{\mathfrak{p}}$ and a fortiori contains $\mathfrak{a} + \mathfrak{b}$. If moreover \mathfrak{a} and \mathfrak{b} are finitely generated and D is a Prüfer domain, then $\mathfrak{a} + \mathfrak{b}$ is finitely generated, hence invertible, thus divisorial and it is closed. \diamond

It is worth giving now an example of a (one-dimensional Noetherian) domain where the closure of the union of two ideals is not an ideal.

Example 3.13 Let k be a field of characteristic $p \neq 2$ and $D = k[t^2, t^3]$ the ring of polynomials with no terms in t . Let $f = \frac{X^2}{t^4}$. It is clear that f is integer-valued on t^2D and on t^3D , thus on the union of these two principal ideals. However $f(t^2 + t^3) = 1 + 2t + t^2$ is not in D .

About the closure of the union of two ideals we lastly have at least the following, whatever the domain D .

Corollary 3.14 *Let \mathfrak{a} and \mathfrak{b} be two coprime ideals of D (that is, $\mathfrak{a} + \mathfrak{b} = D$). Then $\overline{\mathfrak{a} \cup \mathfrak{b}} = D$.*

Proof. For each maximal ideal \mathfrak{m} , the $D_{\mathfrak{m}}$ -closure of the union $\mathfrak{a} \cup \mathfrak{b}$ at \mathfrak{m} clearly contains both $\mathfrak{a}_{\mathfrak{m}}$ and $\mathfrak{b}_{\mathfrak{m}}$. Since $D_{\mathfrak{m}}$ is a local ring and $\mathfrak{a}_{\mathfrak{m}} + \mathfrak{b}_{\mathfrak{m}} = D_{\mathfrak{m}}$, one of the ideals $\mathfrak{a}_{\mathfrak{m}}$ and $\mathfrak{b}_{\mathfrak{m}}$ must be equal to $D_{\mathfrak{m}}$, thus the local closure contains $D_{\mathfrak{m}}$. From Proposition 3.5, $\overline{\mathfrak{a} \cup \mathfrak{b}}$ contains the intersection of the local closures, thus it contains D . \diamond

4 Topological and polynomial closure

Recall a *Zariski ring* is a Noetherian ring R equipped with an \mathfrak{a} -adic topology, such that each ideal is topologically closed. Equivalently \mathfrak{a} is contained in the Jacobson radical of R (e.g. see [19, Theorem 56]). For a Zariski domain, we now relate the polynomial closure to the topological closure.

Theorem 4.1 *Let D be a Zariski domain. The polynomial D -closure of a fractional subset E of D contains the topologic closure of E .*

Proof. We may as well assume E to be a subset of D [Remark 1.6]. Let $f \in \text{Int}(E, D)$ and x in the topologic closure of E for the \mathfrak{a} -adic topology, we wish to show that $f(x) \in D$. Let $d \in D, d \neq 0$, such that $df \in D[X]$. For each integer n , there is $y \in E$ such that $(x - y) \in \mathfrak{a}^n$. Since $(x - y)$ divides $[df(x) - df(y)]$ in D , then $[df(x) - df(y)] \in \mathfrak{a}^n$. Since $df(y) \in dD$, then $df(x) \in dD + \mathfrak{a}^n$. Therefore $df(x)$ belongs to the topological closure of the ideal dD , which is dD itself, since D is a Zariski ring. Thus, $df(x) \in dD$ and, dividing by d , $f(x) \in D$. \diamond

Corollary 4.2 *Let D be a Zariski domain.*

1. *If E is a topologically dense subset of D , then E is a polynomially dense subset of D .*
2. *If E is a polynomially closed fractional subset of D , then E is topologically closed.*

All this applies of course in particular to the \mathfrak{m} -adic topology, if D is a Noetherian local ring, with maximal ideal \mathfrak{m} .

Remark 4.3 If \mathfrak{a} is not contained in the Jacobson radical of D , a dense subset of D for the \mathfrak{a} -adic topology need not be a polynomially dense subset of D and may even be polynomially closed. For example, let $D = k[t]$ be the polynomial ring with coefficients in a field k with q elements (a one-dimensional Noetherian domain). Let \mathfrak{m} be the maximal ideal $\mathfrak{m} = (1 + t)D$ and E be the complement of tD in D . Then E is easily seen to be dense in D for the \mathfrak{m} -adic topology. However E is polynomially closed, indeed $f = \frac{1 - X^{q-1}}{t}$ belongs to $\text{Int}(E, D)$ but $\forall x \notin E, f(x) \notin D$.

If D is a Noetherian local domain, with maximal ideal \mathfrak{m} , integer-valued polynomials are continuous functions from D to D for the \mathfrak{m} -adic topology [9, Proposition 4.3]. If the dimension of D is 1 we show next that each polynomial with coefficients in K is a continuous function from K to K .

Lemma 4.4 *Let D be a one-dimensional Noetherian, local domain, with maximal ideal \mathfrak{m} . Each polynomial f with coefficients in K is uniformly continuous for the \mathfrak{m} -adic topology: there is an integer h such that, for each integer r and each $a, b \in K, (a - b) \in \mathfrak{m}^{r+h} \Rightarrow [f(a) - f(b)] \in \mathfrak{m}^r$.*

Proof. Let d be a non-zero element of D such that df is in $D[X]$. Then, for each a and b in D , $(a - b)$ divides $d[f(a) - f(b)]$ in D . Since D is one-dimensional, Noetherian and local, there is an integer h such that $\mathfrak{m}^h \subseteq dD$. Thus,

$$(a - b) \in \mathfrak{m}^{r+h} \Rightarrow (a - b) \in d\mathfrak{m}^r \Rightarrow d[f(a) - f(b)] \in d\mathfrak{m}^r \Rightarrow [f(a) - f(b)] \in \mathfrak{m}^r. \diamond$$

In this case, we can use a topological argument to prove that the polynomial D -closure of a subset E contains the topological closure of E (for the \mathfrak{m} -adic topology): a polynomial f with coefficients in K is uniformly continuous, hence, if it takes E into D , it takes the (topological) closure of E into the closure of D , but D is obviously topologically closed in K .

Integer-valued polynomials being uniformly continuous functions, they can be considered as continuous functions from the completion \hat{D} of D onto itself. Moreover, we have an analogous of the Stone-Weierstrass theorem if D is analytically irreducible (that is, \hat{D} is a domain), with finite residue field: every continuous function from \hat{D} to \hat{D} can be arbitrarily and uniformly approximated by an integer-valued polynomial [6, p.53] or [10, Theorem 3.3] (this holds in particular if D is a rank-one discrete valuation domain).

Proposition 4.5 *Let D be a local, Noetherian, one-dimensional, analytically irreducible domain with finite residue field and \mathfrak{m} be its maximal ideal. The polynomial and topologic closure (for the \mathfrak{m} -adic topology) of each fractional subset E of K are then equal.*

Proof. From Corollary 4.2, it remains to prove that if E is closed (for the \mathfrak{m} -adic topology), then it is polynomially closed. We may assume E to be a subset of D [Remark 1.6]. We know the polynomial closure \overline{E} of E to be contained in D [Proposition 2.1] and wish to prove that, if $x \in D$ and $x \notin E$, then $x \notin \overline{E}$. Since E is topologically closed and the \mathfrak{m} -adic topology is ultrametric, there is an open and closed neighborhood U of x which does not meet E . Let t be a non-zero element of \mathfrak{m} and n an integer such that $\mathfrak{m}^n \subseteq tD$ (such an integer exists, since D is one-dimensional). From the Stone-Weierstrass theorem, the characteristic function φ of U can be approximated modulo \mathfrak{m}^n by a polynomial $f \in \text{Int}(D)$. Thus $f(x) = 1 + td$, where $d \in D$, whereas $\forall z \in E, f(z) \in tD$. Letting $g = f/t$, then $g \in \text{Int}(E, D)$ but $g(x) \notin D$. \diamond

In particular we recover the fact that the polynomially dense subsets are then the topologically dense subsets of D [3, Proposition 2.1]. Similarly we get that the polynomially closed fractional subsets of D are then the topologically closed fractional subsets of D in this case.

Remark 4.6 In general, polynomially dense subsets are not necessarily topologically dense (and may even be closed for the \mathfrak{m} -adic topology), even if D is a local one-dimensional Noetherian domain. This may happen if D is not analytically irreducible [3, Example 5.3] or if the residue field of D is infinite [3, Remark 3.3].

From Proposition 3.6 we then derive the following characterization of the polynomial closure of a fractional subset in a domain D which is locally analytically irreducible, with finite residue fields. We thus generalize McQuillan's similar result for a Dedekind domain [17, Theorem 2] and recover in particular the characterization we had previously given of the polynomially dense subsets of D [3, p. 206] (as well as Gilmer's original characterization of the polynomially dense subsets of [13, Theorem 2]).

Theorem 4.7 *Let D be a one-dimensional, Noetherian, locally analytically irreducible domain, with finite residue fields, and E a fractional subset of D . The polynomial closure of E is then the intersection of its \mathfrak{m} -adic topological closures in K , where \mathfrak{m} runs over all the maximal ideals of D .*

Proof. We know, by localization properties, that the polynomial closure of E is the intersection of its $\mathfrak{m}D_{\mathfrak{m}}$ -adic topological closures in K , where \mathfrak{m} runs over all the maximal ideals of D . Assuming E to be a subset of D [Remark 1.6], its polynomial closure is contained in D and, for each \mathfrak{m} , we can rather consider the intersection of its $\mathfrak{m}D_{\mathfrak{m}}$ -adic topological closure with D . But this is the topological closure of E for the topology induced by the $\mathfrak{m}D_{\mathfrak{m}}$ -adic topology on D . And this topology is nothing else than the \mathfrak{m} -adic topology. \diamond

Since each ideal of a one-dimensional Noetherian local domain D is clearly topologically closed, we derive in particular the following.

Corollary 4.8 *Let D be a one-dimensional, Noetherian, locally analytically irreducible domain with finite residue fields, then each fractional ideal of D is polynomially closed.*

5 Ring of values

In this section we consider a domain D containing a domain R . We say that D is an extension of R and then set the following definition.

Definition 5.1 Let α be an element of an extension D of R . The set

$$R_{\alpha} = \{f(\alpha) \mid f \in \text{Int}(R)\}.$$

is said to be the *ring of values* of $\text{Int}(R)$ at α .

Clearly the quotient field L of D is a field extension of the quotient field K of R . With these notations, we first record a few obvious facts.

Proposition 5.2 1. *The ring of values R_{α} is a domain.*

2. *$R[\alpha] \subseteq R_{\alpha} \subseteq K[\alpha]$, in particular the quotient field of R_{α} is $K(\alpha)$.*

3. $R_\alpha = R$ if and only if $\alpha \in R$.

Without any assumption on R nor α (α may be algebraic or transcendental) we have the following.

Lemma 5.3 *Let α be an element of an extension D of R .*

1. *If T is a domain such that $R \subseteq T \subseteq D$, if $\alpha \in T$ and if R is a polynomially dense subring of T , then $R_\alpha \subseteq T$.*

2. $\text{Int}(R) \subseteq \text{Int}(R_\alpha)$.

Proof. 1. Let $f \in \text{Int}(R)$. A fortiori $f \in \text{Int}(R, T) = \text{Int}(T)$ (since R is polynomially dense in T). Therefore $f(\alpha) \in T$.

2. Let $f \in \text{Int}(R)$ and $\beta = g(\alpha) \in R_\alpha$, where $g \in \text{Int}(R)$. Then $f(g(X))$ is clearly also in $\text{Int}(R)$, thus $f(\beta) = f(g(\alpha)) \in R_\alpha$. \diamond

Remarks 5.4 1. If R is a subring of T then $\text{Int}(R)$ need not be contained in $\text{Int}(T)$ (we shall give examples below and in the next section). But clearly, if $\text{Int}(R) \subseteq \text{Int}(T)$, then $R_\alpha \subseteq T_\alpha$. However R_α may be contained in T_α (for example if $\alpha \in R$) even if $\text{Int}(R)$ is not contained in $\text{Int}(T)$.

2. It results from the previous lemma and the first remark above (letting $T = R[\alpha]$), that $R_\alpha = R[\alpha]$ if and only if $\text{Int}(R) \subseteq \text{Int}(R[\alpha])$.

Now, if R is a Dedekind domain, Gilbert Gerboud gave several characterizations of the domains D containing R which are such that $\text{Int}(R)$ is contained in $\text{Int}(D)$ [12, theorem 3]. For the sake of completeness we then record the following.

Lemma 5.5 (Gerboud) *Let D be an extension of the Dedekind domain R . The following assertions are equivalent.*

1. *R is a polynomially dense subring of D .*

2. $\text{Int}(R)$ is contained in $\text{Int}(D)$.

3. *For each prime ideal \mathfrak{m} of D such that $\mathfrak{p} = \mathfrak{m} \cap R$ has a finite residue field in R , $\mathfrak{m}D_{\mathfrak{m}} = \mathfrak{p}D_{\mathfrak{m}}$ and $D/\mathfrak{m} \simeq R/\mathfrak{p}$ (in particular each such prime \mathfrak{m} is maximal).*

The characterization in terms of the prime ideals results from localization properties [Corollary 3.7] and various characterizations of (discrete rank-one valuation) domains which are polynomially dense in a larger domain (already in [8] and to be found in [11, proposition 6.1] or [3, Corollaire 2.5]). The restriction to the primes \mathfrak{p} with finite residue fields results from the easy fact that a local ring R with infinite residue field is polynomially dense in every domain D containing

it (clearly then $\text{Int}(R, D) = \text{Int}(D) = D[X]$ and this dates back, (with quite a different terminology) to [5, proposition 3]). On the other hand it is clear that, whatever the domain R , $\text{Int}(R)$ is contained in $\text{Int}(R, D)$, thus if R is polynomially dense in D (that is, $\text{Int}(R, D) = \text{Int}(D)$) then $\text{Int}(R)$ is contained in $\text{Int}(D)$. The converse is more technical and requires R to be a Dedekind domain. Indeed we give below an example where, although R is an *almost Dedekind domain*, $\text{Int}(R)$ is contained in $\text{Int}(D)$ whereas R is not polynomially dense in D .

Example 5.6 Let R be an almost Dedekind domain with at least a maximal ideal \mathfrak{m}_o with finite residue field but such that $R = \bigcap_{\mathfrak{m} \in \mathcal{M}} R_{\mathfrak{m}}$, where \mathcal{M} denotes the set of maximal ideals with infinite residue fields. Then $\text{Int}(R) = R[X]$ [5, Corollaire 3, p. 303], thus $\text{Int}(R)$ is contained in $\text{Int}(D)$, whatever the domain D containing R . However, it results from the previous lemma that there exists a ring D , containing $R_{\mathfrak{m}_o}$ in which $R_{\mathfrak{m}_o}$ is not polynomially dense (for example a discrete rank-one valuation domain such that the ramification index or the residual degree of this extension is greater than 1). A fortiori R is not polynomially dense in D .

From the previous two lemmas we then get immediately the following characterization of R_{α} for a Dedekind domain R .

Theorem 5.7 *Let D be an extension of the Dedekind domain R and $\alpha \in D$. The ring of values R_{α} is then the smallest subring of D containing $R[\alpha]$ in which R is polynomially dense.*

We gave above an example where $\text{Int}(R)$ is contained in $\text{Int}(D)$ but R is not polynomially dense in D . However D was not a ring of values R_{α} of $\text{Int}(R)$ at each α . We thus raise the following question.

Question 5.8 Let α be an element of an extension of a domain R . Is R polynomially dense in R_{α} ?

From Theorem 5.7 we shall recover McQuillan's characterization of R_{α} [18, theorem] in the case where R is a Dedekind domain with finite residue fields and α is algebraic over K . We first note (according to one of McQuillan's arguments) that in this case $\text{Int}(R)$ is a Prüfer domain (for example see [7]) and so is its homomorphic image R_{α} . In particular R_{α} is then integrally closed in its quotient field $K(\alpha)$ and thus contains the integral closure of R in $K(\alpha)$. From the previous theorem we thus get the following.

Corollary 5.9 *Let R be a Dedekind domain with finite residue fields, α an element of an extension of its quotient field K and S the integral closure of R in $K(\alpha)$. Then R_{α} is the smallest overring of S containing α in which R is polynomially dense.*

Remark 5.10 In general R_α need not be integrally closed, even if R is integrally closed. For example if R is a Dedekind domain *with infinite residue fields*, then $\text{Int}(R) = R[X]$, thus $R_\alpha = R[\alpha]$ and one may choose α in an algebraic extension of K such that $R[\alpha]$ is not integrally closed.

We did not restrict ourselves to the case where α is algebraic in the previous corollary. In fact, if α is transcendental then R_α is clearly isomorphic to $\text{Int}R$ and without any assumption on R , we indeed have the following.

Proposition 5.11 *Let R be an infinite domain then $\text{Int}(R)$ is the smallest overring of $R[X]$ in which R is polynomially dense.*

Proof. From Lemma 5.3, $\text{Int}(R)$ is contained in every overring of $R[X]$ in which R is polynomially dense. We want to prove that R is polynomially dense in $\text{Int}(R)$ itself. Let $f \in \text{Int}(R, \text{Int}(R))$, and write $f = g_0 + g_1T + \dots + g_nT^n$ as a polynomial in T with coefficients in the quotient field $K(X)$ of $\text{Int}(R)$. Consider $n+1$ distinct elements r_0, r_1, \dots, r_n of R , the values $f(r_i)$ of f on these elements are in $\text{Int}(R)$ and a fortiori in $K[X]$. We thus get a system, whose Vandermonde determinant $d = \prod_{i < j} (r_j - r_i)$ is invertible in K . Hence the coefficients of f are in fact in $K[X]$. Now, let $g \in \text{Int}(R)$, then $f(g)$ is an element of $K[X]$ and, $\forall r \in R, f(g)(r) = f(g(r)) \in R$ (since $g(r) \in R$), thus $f \in \text{Int}(\text{Int}(R))$. \diamond

If R is a Dedekind domain and α is algebraic over its quotient field K , it results from the Krull-Akizuki theorem that S is a Dedekind domain (whether α is separable or not). Thus each overring T of S is determined by a set \mathcal{X}_T of primes of S (namely the primes that lift in T). If the residue fields of R are finite, it results from Lemma 5.5 that R is polynomially dense in T if and only if each prime \mathfrak{m} in \mathcal{X}_T is such that, letting $\mathfrak{p} = \mathfrak{m} \cap R$, the ramification index $e(\mathfrak{m}/\mathfrak{p})$ and the residual degree $f(\mathfrak{m}/\mathfrak{p})$ are both equal to 1. The primes of S satisfying this condition form a set Π_1 (and are called the *split primes* of S). On the other hand an overring T of S contains α , if and only if each prime \mathfrak{m} in \mathcal{X}_T is such that $v_{\mathfrak{m}}(\alpha) \geq 0$ (where $v_{\mathfrak{m}}$ is the valuation associated to \mathfrak{m}). Let Π_α be the subset of Π_1 formed by all the split primes of S which satisfy this second condition. The larger is the set \mathcal{X}_T that determines T , the smaller is the overring T , hence the smallest overring T of S , containing α , in which R is polynomially dense is determined by Π_α . From Corollary 5.9 we thus recover McQuillan's result.

Corollary 5.12 (D. McQuillan) *Let R be a Dedekind domain with quotient field K and finite residue fields, $L = K(\alpha)$ an algebraic extension of K and S the integral closure of R in L . The ring of values R_α is then the overring of S determined by Π_α . If in particular $\alpha \in S$, then R_α is determined by Π_1 and so is independent of the choice of α in S .*

6 Inclusion of $\text{Int}(D)$ in $\text{Int}(D')$

If D is a domain and D' its integral closure, $\text{Int}(D)$ need not be contained in $\text{Int}(D')$. It is however obvious that it is contained in $\text{Int}(D, D')$ and R.Gilmer at al. [14, Proposition 2.2] have shown that, if D is Noetherian, then $\text{Int}(D, D')$ is the integral closure of $\text{Int}(D)$. They characterized also the *one-dimensional* Noetherian domains such that $\text{Int}(D)$ is contained in $\text{Int}(D')$ [14, Proposition 6.1]. We generalize here this result, whatever the dimension of D , proving in particular that, equivalently, D is polynomially dense in D' . Now, from the *Mori-Nagata integral closure theorem* [19, p.264], D' is a Krull domain (although not necessarily Noetherian). We thus first record the following characterization of the polynomially dense subrings of a Krull domain, which is easily derived from the local properties of the polynomially dense subsets of a Krull domain [Corollary 3.7], and the characterization of the dense subrings of a discrete rank-one valuation domain (e.g. [3, Corollaire 2.5]).

Lemma 6.1 *Let D be a Krull domain and R a subring of D . The following assertions are equivalent.*

1. R is a polynomially dense subring of D .
2. For each height-one prime \mathfrak{q} of D , letting $\mathfrak{p} = \mathfrak{q} \cap R$, if D/\mathfrak{q} is finite, then $R/\mathfrak{p} = D/\mathfrak{q}$ and $\mathfrak{p}D\mathfrak{q} = \mathfrak{q}D\mathfrak{q}$ and if D/\mathfrak{q} is infinite, then R/\mathfrak{p} is infinite.

Now comes the main theorem of this section.

Theorem 6.2 *Let D be a Noetherian domain. The following assertions are equivalent.*

1. $\text{Int}(D)$ is contained in $\text{Int}(D')$.
2. D is a polynomially dense subset of D' .
3. For each height-one prime \mathfrak{q} of D' with finite residue field, letting $\mathfrak{p} = \mathfrak{q} \cap D$, then $D/\mathfrak{p} \simeq D'/\mathfrak{q}$ and $\mathfrak{p}D'\mathfrak{q} = \mathfrak{q}D'\mathfrak{q}$.

Proof. We first prove the first two assertions to be equivalent: if D is a polynomially dense subset of D' , then $\text{Int}(D') = \text{Int}(D, D')$ contains $\text{Int}(D)$. Conversely, if $\text{Int}(D) \subseteq \text{Int}(D')$, then $\text{Int}(D') = \text{Int}(D, D')$, since $\text{Int}(D')$ is integrally closed [4, Proposition 2.1] and contained in $\text{Int}(D, D')$, which is the integral closure of $\text{Int}(D)$.

In the previous lemma we characterized the polynomially dense subrings of a Krull domain. The condition on an height-one prime \mathfrak{q} with finite residue field is exactly the last assertion above and the condition on an height-one prime \mathfrak{q} with infinite residue field is always satisfied since, from the *Mori-Nagata integral closure theorem*, if D'/\mathfrak{q} is infinite, then D/\mathfrak{p} is infinite. \diamond

If D is not Noetherian, then $\text{Int}(D, D')$ need not be the integral closure of $\text{Int}(D)$, according to the next example, which stems from a classical pullback construction.

Example 6.3 Let q be the finite field with q elements, ${}_q$ its algebraic closure, $V = {}_q[[t]]$ and $D = q + t_q[[T]]$. Then D is a pseudo-valuation one-dimensional domain [15] sharing its maximal ideal $\mathfrak{m} = t_q[[t]]$ with its integral closure $D' = V$. We prove now that $\text{Int}(D, D')$ is not integral over $\text{Int}(D)$. It is easy to check that the polynomial $f = \frac{X^q - X}{t}$ is in $\text{Int}(D, D')$. Now, let $a \in q$, then $f(at) \equiv -a \pmod{\mathfrak{m}}$ in D' . If f were a root of a monic polynomial of degree n over $\text{Int}(D)$, every element of ${}_q$ would then be of degree n over q .

Considering more generally a subset E of the quotient field K of D , we had shown that $\text{Int}(E, D')$ is the integral closure of $\text{Int}(E, D)$, if D is a *local* one-dimensional Noetherian domain with finite residue field [4, Proposition 2.3]. From Proposition 3.1, this local result may easily be globalized, hence the same holds if D is a one-dimensional Noetherian domain with finite residue fields. It can even more generally be seen that, if D is a one-dimensional Noetherian domain and E is a subset of K such that, for each maximal ideal \mathfrak{m}' of D' with infinite residue field, E contains infinitely many classes modulo \mathfrak{m}' , then $\text{Int}(E, D')$ is the integral closure of $\text{Int}(E, D)$ [4, Remark 2.4]. We lastly give an example of a (pseudo-valuation) one-dimensional Noetherian domain D (with infinite residue field) and a subset E of D , such that $\text{Int}(E, D')$ is *not* the integral closure of $\text{Int}(E, D)$.

Example 6.4 Let $V = [[t]]$, the ring of power series with coefficients in the field of complex numbers, and $D = \mathbb{C} + t[[t]]$. Then D is a pseudo-valuation one-dimensional Noetherian domain sharing its maximal ideal $\mathfrak{m} = t[[t]]$ with its integral closure $D' = V$. Since $D/\mathfrak{m} \simeq \mathbb{C}$ is infinite, then $\text{Int}(D') = D'[X]$ and $\text{Int}(\mathfrak{m}, D') = \text{Int}(tD', D') = D'[X/t]$.

Now we determine $\text{Int}(\mathfrak{m}, D)$. Let $f \in \text{Int}(\mathfrak{m}, D)$, a fortiori $f(tX) \in \text{Int}(D)$, thus $f \in D[X/t]$. Write $f = \sum_{i=1}^n a_i (X/t)^i$, where $a_i \in D$. For each $z \in \mathbb{C}$, $f(z) \in D$. If some coefficient a_i were not in \mathfrak{m} , for $i > 0$, there would be a non constant real polynomial taking real values on every complex number. Thus $\text{Int}(\mathfrak{m}, D) = D + \mathfrak{m}D[X/t]$.

We finally determine the integral closure of $\text{Int}(\mathfrak{m}, D)$. Let $Y = X/t$, then $\text{Int}(\mathfrak{m}, D) = D + \mathfrak{m}D[Y]$. The ring $R = D' + \mathfrak{m}D'[Y]$ shares the ideal $\mathfrak{m}D'[Y]$ with the ring of polynomials $D'[Y]$, which is integrally closed. The quotient $R/\mathfrak{m}D'[Y] \simeq D'/\mathfrak{m}$ is integrally closed in $D'/\mathfrak{m}D'[Y] \simeq (D'/\mathfrak{m})[Y]$, thus R is integrally closed (in its quotient field) [2, Proposition 2]. On the other hand, R is clearly integral over $\text{Int}(\mathfrak{m}, D) = D + \mathfrak{m}D[Y]$. Hence the integral closure of $\text{Int}(\mathfrak{m}, D)$ is the ring $R = D' + \mathfrak{m}D'[X/t]$, which is strictly contained in $\text{Int}(\mathfrak{m}, D') = D'[X/t]$.

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