INTEGER-VALUED POLYNOMIALS AND PRÜFER v-MULTIPLICATION DOMAINS

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Abstract. Let D be a domain with quotient field K. We consider the ring $\mathrm{Int}(D) := \{f \in K[X]; f(D) \subseteq D\}$ of *integer-valued polynomial ring over D.* We completely characterize the domains D for which $\mathrm{Int}(D)$ is a *Prüfer v-multiplication domain.*

Introduction

Given a domain D and a nonzero fractional ideal I, recall that the v-closure of I is the ideal $I_v := (I^{-1})^{-1} = (D\colon (D\colon I))$ and its t-closure the union $I_t := \bigcup J_v$, where J runs over the finitely generated ideals contained in I. A v-ideal (or *divisorial* ideal) is a nonzero ideal such that $I = I_v$ and a v-finite ideal is such that $I_v = J_v$, where J is of finite type. A t-ideal is a nonzero ideal such that $I = I_t$. A *t-prime* ideal is a prime ideal which is also a t-ideal, and a *t-maximal* ideal is a maximal element among the integral t-ideals. It is known that a t-maximal ideal is prime and that each integral t-ideal is contained in a t-maximal ideal [10, Corollaire 1 & 2, p.30]. Recall that a *Prüfer v-multiplication domain* (PvMD) is a domain in which the v-finite ideals form a group [6]. This is equivalent to say that the localization at each t-prime ideal is a valuation domain [6, Theorem 5]. Thus PvMDs generalize Prüfer domains (for which, among several definitions, the localization at each prime ideal is a valuation domain). For the basic properties of these rings we refer the reader also to [11] and [12].

In this paper, we focus on the ring of integer-valued polynomials of a domain D (with quotient field K), that is,

$$\mathrm{Int}(D) := \{f \in K[X] \mid f(D) \subseteq D\},$$

and we characterize the domains D for which $\mathrm{Int}(D)$ is a PvMD.

We make a distinction between two types of prime ideals: on one side, the primes $\mathfrak{p}$ such that $\mathrm{Int}(D) \not\subseteq D_{\mathfrak{p}}[X]$ (necessarily maximal with finite residue field [1, Proposition I.3.4]), that we call the *int primes*, and on the other, the primes $\mathfrak{p}$ such that $\mathrm{Int}(D) \subseteq D_{\mathfrak{p}}[X]$, that we call the *polynomial primes.* In a sense, the behaviour of $\mathrm{Int}(D)$ above a polynomial prime is similar to the behaviour of the polynomial ring $D[X]$, whereas we can consider an int prime as a true "integer-valued" prime ideal. Note that $\mathrm{Int}(D) \subseteq D_{\mathfrak{p}}[X]$ (that is, $\mathfrak{p}$ is polynomial) is equivalent to $\mathrm{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ (since obviously $\mathrm{Int}(D)$ contains $D[X]$), but does not imply $\mathrm{Int}(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ (see [Example 5.3] below).

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In a first section, we give a necessary condition on the int prime ideals: if $\mathrm{Int}(D)$ is a PvMD, then each int prime ideal $\mathfrak{m}$ is an height-one (maximal) ideal. Since D is then itself a PvMD, we show that it follows that $D_{\mathfrak{m}}$ is a rank-one discrete valuation domain.

In a second section, in order to give another necessary condition, we first develop some generalities on ideals defined by a filter $\mathcal{U}$ on a family $\{\mathfrak{a}_\lambda\}$ of ideals; this is one of our main tools. Using this construction we show that, if $\mathrm{Int}(D)$ is a PvMD, then each nonzero polynomial prime ideal of D contains a finitely generated ideal which is not contained in any int prime ideal.

In a third section we prove that these conditions are sufficient and obtain our main result: $\mathrm{Int}(D)$ is a PvMD, if and only if

(a) D is a PvMD,
(b) each int prime ideal of D is an height-one (maximal) ideal,
(c) each nonzero polynomial t-prime ideal of D contains a finitely generated ideal which is not contained in any int prime ideal.

Under these conditions, we show that, if $\mathfrak{P}$ is a prime t-ideal of $\mathrm{Int}(D)$, then $\mathrm{Int}(D)_{\mathfrak{P}}$ is a valuation domain. Letting $\mathfrak{p} = \mathfrak{P} \cap D$, we consider separately the case where $\mathfrak{p}$ is polynomial and where it is not. In fact, assuming (a) (that is, D is a PvMD), we can conclude under condition (b) for the int case, and under condition (c) for the polynomial case.

In the fourth section, we consider the partition of $\mathrm{Spec}(D)$ into the subsets Δ_0 of int prime ideals and Δ_1 of polynomial prime ideals. Letting $D_0 := \bigcap_{\mathfrak{m}\in\Delta_0} D_{\mathfrak{m}}$ and $D_1 := \bigcap_{\mathfrak{p}\in\Delta_1} D_{\mathfrak{p}}$, we show that $\mathrm{Int}(D) = \mathrm{Int}(D_0) \cap D_1[X]$. If $\mathrm{Int}(D)$ is a PvMD, we show that Δ_0 satisfies a double boundedness property, from which we derive that $\mathrm{Int}(D_0)$ is a Prüfer domain.

In the last section we give two examples, in the special case where D is an almost Dedekind domain. In the first one, $\mathrm{Int}(D_0)$ is a Prüfer domain, yet $\mathrm{Int}(D)$ is not a PvMD. In the second one, $\mathrm{Int}(D)$ is a PvMD and there exists a polynomial prime $\mathfrak{p}$ of D, such that $\mathrm{Int}(D_{\mathfrak{p}}) \neq D_{\mathfrak{p}}[X]$.

1. Necessary conditions and int prime ideals

We first focus on the int prime ideals of D, that is, the prime ideals $\mathfrak{m}$ such that $\mathrm{Int}(D) \not\subseteq D_{\mathfrak{m}}[X]$. We first reword a result of D. Rush [15, Theorem 1.5] & [1, Exercise I.25]. If x is an element of the quotient field K of D, we denote by $(D :_D x)$ the conductor of x, that is, $(D :_D x) = \{\alpha \in D \mid \alpha x \in D\}$.

Lemma 1.1. *Let D be a domain with quotient field K and B be an overring of D. The following assertions are equivalent:*

(i) $\mathrm{Int}(D) \not\subseteq B[X]$,
(ii) *there exists $x \in K$, $x \notin B$, and an integer n such that, for each prime ideal $\mathfrak{p}$ of D containing $(D :_D x)$, we have the double boundedness condition $|D/\mathfrak{p}| \leq n$, and $x\alpha^n \in D_{\mathfrak{p}}$, for $\alpha \in \mathfrak{p}$,*
(iii) *there exists $x \in K$, $x \notin B$, and an integer-valued polynomial $f \in \mathrm{Int}(D)$ of the form $f = xX^n(X^m - 1)^n$.*

Proof. (i) ⇒ (ii) By hypothesis, there exists a polynomial $f \in \mathrm{Int}(D)$ with at least one coefficient $x \notin B$. Let d be the degree of f. If $a_0, \ldots, a_d$ are $d+1$ elements arbitrarily chosen in D, then $f(a_i) \in D$, for $0 \leq i \leq d$. We may conclude from a standard Vandermonde argument that we have $x\prod_{0\leq i<j\leq d}(a_j - a_i) \in D$ [1,

Proposition I.3.1]. For each prime ideal $\mathfrak{p}$ containing $(D :_D x)$, we then have $|D/\mathfrak{p}| \leq d$. For $\alpha \in \mathfrak{p}$, choose $a_i = \alpha^i$, we then have $x \prod_{0 \leq i < j \leq d} (\alpha^j - \alpha^i) \in D$. We can rewrite $\prod_{0 \leq i < j \leq d} (\alpha^j - \alpha^i)$ as the product of α^e with factors of the form $(\alpha^t - 1)$ (where e is independent from α and $\mathfrak{p}$). If $\alpha \in \mathfrak{p}$, then $(\alpha^t - 1) \notin \mathfrak{p}$, and we derive that $\alpha^e x \in D_{\mathfrak{p}}$.
(ii) ⇒ (iii) Since the cardinal of the residue fields is bounded, there is an integer q such that the polynomial $f = x(X^q - X)^n$ is integer-valued. Indeed, for each prime $\mathfrak{p}$ containing $(D :_D x)$, $X^q - X$ takes its values in $\mathfrak{p}$, thus $f(D) \subseteq D_{\mathfrak{p}}$, and for each prime $\mathfrak{p}$ that does not contain $(D :_D x)$, $f \in D_{\mathfrak{p}}[X]$, since $x \in D_{\mathfrak{p}}$, and a fortiori $f(D) \subseteq D_{\mathfrak{p}}$.
(iii) ⇒ (i) Obvious. □

If $\mathfrak{m}$ is an int prime ideal, that is, $\mathrm{Int}(D) \not\subseteq D_{\mathfrak{m}}[X]$, it follows from the previous lemma that there is an element $x \notin D_{\mathfrak{m}}$ such that every prime containing $(D :_D x)$ has a finite residue field. Thus, every such prime is a minimal prime ideal of $(D :_D x)$. In particular, this is the case of the ideal $\mathfrak{m}$ itself, which contains $(D :_D x)$, since $x \notin D_{\mathfrak{m}}$ (see also [1, Proposition I.3.4]). Since a conductor ideal is a divisorial ideal, a fortiori a t-ideal, we derive immediately the following.

Proposition 1.2. *If $\mathfrak{m}$ is an int prime ideal of the domain D, then $\mathfrak{m}$ is a t-ideal.*

If D is a PvMD, it follows that $D_{\mathfrak{m}}$ is a valuation domain, and since $\mathrm{Int}(D) \not\subseteq D_{\mathfrak{m}}[X]$, the maximal ideal of this valuation domain is principal and its residue field is finite [1, Proposition I.3.16]. We thus derive the following:

Corollary 1.3. *Let D be a PvMD. Then, for each int prime ideal $\mathfrak{m}$ of D, $D_{\mathfrak{m}}$ is a valuation domain with finite residue field, and its maximal ideal is principal.*

Considering a maximal ideal $\mathfrak{M}$ of $\mathrm{Int}(D)$ above an int prime ideal $\mathfrak{m}$ of D, we show that, under certain conditions, $\mathfrak{M}$ is itself an int prime ideal of $\mathrm{Int}(D)$. Namely, if $\mathfrak{M}$ contains the ideal $\mathrm{Int}(D, \mathfrak{m}) := \{f \in \mathrm{Int}(D) \mid f(D) \subseteq \mathfrak{m}\}$, and thus, in particular, if $\mathfrak{M}$ is of the form $\mathfrak{M}_a := \{f \in \mathrm{Int}(D) \mid f(a) \in \mathfrak{m}\}$, for some $a \in D$.

Proposition 1.4. *Let $\mathfrak{m}$ be an int prime ideal of a domain D and $\mathfrak{M}$ be a prime ideal of $\mathrm{Int}(D)$ containing the ideal $\mathrm{Int}(D, \mathfrak{m})$. Then $\mathfrak{M}$ is an int prime ideal of $\mathrm{Int}(D)$. In particular, $\mathfrak{M}$ is maximal and it is a t-ideal of $\mathrm{Int}(D)$.*

Proof. By hypothesis, there is a polynomial $f \in \mathrm{Int}(D)$ with some coefficient x which is not in $D_{\mathfrak{m}}$. For each $g \in \mathrm{Int}(D)$, we clearly have $f(g) \in \mathrm{Int}(D)$, in other words, $f \in \mathrm{Int}\big(\mathrm{Int}(D)\big)$. We claim that x does not belong to $\mathrm{Int}(D)_{\mathfrak{M}}$ (and hence that $\mathrm{Int}\big(\mathrm{Int}(D)\big) \not\subseteq \mathrm{Int}(D)_{\mathfrak{M}}[X]$). Indeed, suppose, by way of contradiction, that $hx \in \mathrm{Int}(D)$, where $h \in \mathrm{Int}(D), h \notin \mathfrak{M}$. A fortiori, $h \notin \mathrm{Int}(D, \mathfrak{m})$, hence, there is $a \in D$, such that $h(a) \notin \mathfrak{m}$. We reach a contradiction, since we would then have $h(a)x \in D$, that is, $x \in D_{\mathfrak{m}}$. □

Remarks 1.5. (1) It may occur that there is a chain of prime ideals of length $n \geq 2$ in $\mathrm{Int}(D)$ above some maximal ideal $\mathfrak{m}$ of D [1, Example V.4.3]. Clearly, $\mathfrak{m}$ must then be an int prime ideal, whereas some prime ideals of $\mathrm{Int}(D)$ above $\mathfrak{m}$ are not maximal, and thus, are not int prime ideals of $\mathrm{Int}(D)$.
(2) If we assume that, for some int prime ideal $\mathfrak{m}$, $D_{\mathfrak{m}}$ is a valuation domain, then every ideal $\mathfrak{M}$ of $\mathrm{Int}(D)$ above $\mathfrak{m}$ is an int prime. Indeed, there is a polynomial $f \in \mathrm{Int}(D)$ with some coefficient x which is not in $D_{\mathfrak{m}}$. As above, $f \in \mathrm{Int}\big(\mathrm{Int}(D)\big)$. In this situation we conclude that $x \notin \mathrm{Int}(D)_{\mathfrak{M}}$ from the fact that $x^{-1} \in \mathfrak{m}D_{\mathfrak{m}}$.

We reach our first necessary condition, and before, we state a lemma, the proof of which is immediate and left to the reader.

Lemma 1.6. *Let D be a domain, $\mathfrak{p}$ be a nonzero prime ideal of D, and $a \in D$. The localization of $D[X]$ with respect to the prime ideal $(\mathfrak{p}, X - a)$ is not a valuation domain.*

Proposition 1.7. *Let D be a domain such that* $\mathrm{Int}(D)$ *is a PvMD. Then every int prime ideal of D is an height-one prime ideal.*

Proof. Let $\mathfrak{m}$ be an int prime ideal of D. It follows from Proposition 1.4 that $\mathfrak{M}_0 := \{f \in \mathrm{Int}(D) \mid f(0) \in \mathfrak{m}\}$ is a t-ideal of $\mathrm{Int}(D)$. Hence $\mathrm{Int}(D)_{\mathfrak{M}_0}$ is a valuation domain. Suppose, by way of contradiction, that $\mathfrak{m}$ contains some nonzero prime ideal $\mathfrak{p}$. Then $\mathfrak{M}_0$ contains the prime ideal $\mathfrak{P}_0 := \{f \in \mathrm{Int}(D) \mid f(0) \in \mathfrak{p}\}$. The localization $\mathrm{Int}(D)_{\mathfrak{P}_0}$ would be a valuation domain. But $\mathrm{Int}(D) \subseteq D_{\mathfrak{p}}[X]$, hence $\mathrm{Int}(D)_{\mathfrak{P}_0}$ is also the localization of $D[X]$ with respect to the prime ideal $(\mathfrak{p}, X)$ (upper to $\mathfrak{p}$). We reach a contradiction, according to Lemma 1.6. □

Corollary 1.8. *Let D be a domain such that* $\mathrm{Int}(D)$ *is a PvMD. Then, for each int prime ideal $\mathfrak{m}$ of D, $D_{\mathfrak{m}}$ is a discrete rank-one valuation domain with finite residue field.*

Proof. Choose $a \in D$, and let $\mathfrak{M}_a := \{f \in \mathrm{Int}(D) \mid f(a) \in \mathfrak{m}\}$. It follows from Proposition 1.4 that $\mathfrak{M}_a$ is an int prime ideal of $\mathrm{Int}(D)$. Thus $\mathrm{Int}(D)_{\mathfrak{M}_a}$ is a valuation domain. We claim that $D_{\mathfrak{m}} = \mathrm{Int}(D)_{\mathfrak{M}_a} \cap K$. Clearly $D_{\mathfrak{m}}$ is contained in $\mathrm{Int}(D)_{\mathfrak{M}_a} \cap K$, and conversely, if $x \in K$ belongs to $\mathrm{Int}(D)_{\mathfrak{M}_a}$, there is a polynomial $f \in \mathrm{Int}(D)$, $f \notin \mathfrak{M}_a$, such that $fx \in \mathrm{Int}(D)$. In particular, we have $f(a)x \in D$, and hence, $x \in D_{\mathfrak{m}}$, as $f(a) \notin \mathfrak{m}$. Hence $D_{\mathfrak{m}}$ is a valuation domain. Since $\mathrm{Int}(D) \nsubseteq D_{\mathfrak{m}}[X]$, the maximal ideal of this valuation domain is principal and its residue field is finite [1, Proposition I.3.16]. From the previous proposition, $\mathfrak{m}$ is an height-one prime ideal, and hence, $D_{\mathfrak{m}}$ is a discrete rank-one valuation domain. □

If $\mathrm{Int}(D)$ is a PvMD, it is known that D itself is a PvMD. [16, Propositions 2.1 & 3.1]. This is our second necessary condition. For sake of completeness, we give a complete proof of this result, and first establish a general lemma:

Lemma 1.9. *Let B be a domain with quotient field L, let K be a subfield of L, and let $D := B \cap K$. Then every t-ideal of D is contained in a t-ideal of B.*

Proof. Let I be a t-ideal of D. Assume, by way of contradiction, that the t-closure of IB (in B) is B: there is a finitely generated ideal J, contained in I, such that the v-closure of JB (in B again) is B. Equivalently the inverse of JB, that is, the conductor $(B\colon JB)$, is B. Thus we may find finitely many elements $a_1, \ldots, a_n$ in I (the generators of J) such that

$$(B\colon JB) = a_1^{-1}B \cap \ldots \cap a_n^{-1}B = B.$$

Intersecting with K, we obtain

$$(D\colon J) = a_1^{-1}D \cap \ldots \cap a_n^{-1}D = (a_1^{-1}B \cap K) \cap \ldots \cap (a_n^{-1}B \cap K) = D.$$

And hence, the v-closure of J (in D) is D. This contradicts the fact that I is a t-ideal. □

Proposition 1.10. *Let D be a domain such that* $\mathrm{Int}(D)$ *is a PvMD, then D is a PvMD.*

Proof. Let $\mathfrak{p}$ be a t-prime ideal of D. From the previous lemma, $\mathfrak{p}$ is contained in a t-ideal $\mathfrak{Q}$ of $\mathrm{Int}(D)$. We let $\mathfrak{q} := D \cap \mathfrak{Q}$. We claim that $D_\mathfrak{q}$ is a valuation domain, from which it follows that $D_\mathfrak{p}$ is itself a valuation domain, since it is an overring of $D_\mathfrak{q}$. If $\mathfrak{q}$ is an int prime ideal of D, our claim follows from Corollary 1.8. If $\mathfrak{q}$ is a polynomial prime, then $D_\mathfrak{q}[X] = \mathrm{Int}(D)_\mathfrak{q}$, and thus $\mathrm{Int}(D)_\mathfrak{Q} = D[X]_{\mathfrak{Q}'}$, where $\mathfrak{Q}' = \mathfrak{Q} \cap D[X]$. Since $\mathfrak{Q}$ is a t-ideal and $\mathrm{Int}(D)$ is a PvMD, it follows that $D[X]_{\mathfrak{Q}'}$ is a valuation domain. To reach our conclusion, we finally verify that $D_\mathfrak{q} = D[X]_{\mathfrak{Q}'} \cap K$. One containment is obvious, and conversely, if $x \in D[X]_{\mathfrak{Q}'} \cap K$, there is a polynomial $f \in D[X]$, $f \notin \mathfrak{Q}'$, such that $fx \in D[X]$. Since the extended prime $\mathfrak{q}[X]$ is contained in $\mathfrak{Q}'$, at least one coefficient a of f does not belong to $\mathfrak{q}$, and hence, we have $ax \in D$, that is, $x \in D_\mathfrak{q}$. □

2. Limit ideal with respect to a filter

Let $\{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals of a domain D. For each $x \in D$, we let $B(x)$ be the subfamily of ideals containing x, in fact, we rather consider $B(x)$ as the corresponding set of indices, that is, $B(x) := \{\lambda \mid x \in \mathfrak{a}_\lambda\}$. One can check, for each x and y in D, the following containments:

- $B(x) \cap B(y) \subseteq B(x + y)$,
- $B(x) \cup B(y) \subseteq B(xy)$.

More generally, if J is an ideal, we let $B(J)$ be the subfamily of ideals containing J. We shall use the following result:

Lemma 2.1. *Let $J = (x_1, \dots, x_n)$ be a finitely generated ideal. Then*

$$\bigcap_{i=1}^{n} B(x_i) = \{\lambda \mid J \subseteq \mathfrak{a}_\lambda\} = B(J).$$

□

Proof. Obviously $\mathfrak{a}_\lambda$ contains each x_i if and only if it contains J, since J is generated by the elements $\{x_1, \dots, x_n\}$. □

Next, we consider a filter $\mathcal{U}$ over Λ, and set

$$\mathfrak{a}_\mathcal{U} := \{x \in D \mid B(x) \in \mathcal{U}\}.$$

More explicitly, we have that

$$\mathfrak{a}_\mathcal{U} = \bigcup_{B\in\mathcal{U}} \Big(\bigcap_{\lambda\in B} \mathfrak{a}_\lambda\Big).$$

It is clear from the above properties of $B(x + y)$ and $B(xy)$, that $\mathfrak{a}_\mathcal{U}$ is an ideal of D (which may be reduced to (0)). We say that $\mathfrak{a}_\mathcal{U}$ is *the limit of the ideals* $\{\mathfrak{a}_\lambda\}$, *with respect to the filter* $\mathcal{U}$.

We may conversely ask if a given ideal I is the limit of a family $\{\mathfrak{a}_\lambda\}$ of ideals with respect to some filter $\mathcal{U}$, or at least if I is contained in such a limit ideal, that is, if the sets $B(x)$, for $x \in I$ belong to a filter $\mathcal{U}$. We leave the following to the reader (using Lemma 2.1).

Lemma 2.2. *Let $\{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals. The following assertions are equivalent, for an ideal I of D:*

(1) *I is contained in the limit ideal $\mathfrak{a}_{\mathcal{U}}$ of the family $\{\mathfrak{a}_\lambda\}$, with respect to some filter $\mathcal{U}$,*
(2) *the finite intersections of sets of the form $B(x)$, for $x \in I$, are not empty,*
(3) *for each finitely generated ideal J contained in I, there exists some ideal $\mathfrak{a}_\lambda$ containing J.*

Remark 2.3. Given an ideal I satisfying the hypotheses of Lemma 2.2, the filter $\mathcal{U}$ can be taken to be the family of subsets $\{W \subseteq \Lambda \mid B(J) \subseteq W$, for some finitely generated ideal $J \subseteq I\}$.

We next relate some properties of the ideals $\{\mathfrak{a}_\lambda\}$ to properties of their limit $\mathfrak{a}_{\mathcal{U}}$. For instance, recalling that every filter is contained in an ultrafilter, we leave again the following to the reader.

Lemma 2.4. *The limit of a family of prime ideals, with respect to an ultrafilter, is a prime ideal.*

We are more particularly interested in relating properties of t-ideals to their limit:

Proposition 2.5. *Let $\mathfrak{a}_{\mathcal{U}}$ be the limit of a family $\{\mathfrak{a}_\lambda\}$ of t-ideals, with respect to a filter $\mathcal{U}$. If $\mathfrak{a}_{\mathcal{U}}$ is a nonzero ideal, then $\mathfrak{a}_{\mathcal{U}}$ is a t-ideal.*

Proof. If a nonzero finitely generated ideal J is contained in $\mathfrak{a}_{\mathcal{U}}$, we want to show that J_v is also contained in $\mathfrak{a}_{\mathcal{U}}$. Since, each $\mathfrak{a}_\lambda$ is a t-ideal, we have $B(J_v) = B(J)$ ($\mathfrak{a}_\lambda$ contains J if and only if it contains J_v). On the other hand, from Lemma 2.1 and filter properties, we have $B(J) \in \mathcal{U}$. Hence, $B(J_v) \in \mathcal{U}$. A fortiori $B(x) \in \mathcal{U}$, for each $x \in J_v$, that is, $x \in \mathfrak{a}_{\mathcal{U}}$, for each $x \in J_v$. □

We have a similar result for the limit of int prime ideals.

Lemma 2.6. *Let D be a domain such that* Int(D) *is a PvMD. Let $\mathfrak{m}_{\mathcal{U}}$ be the limit of a family $\{\mathfrak{m}_\lambda\}$ of int prime ideals with respect to an ultrafilter $\mathcal{U}$. If $\mathfrak{m}_{\mathcal{U}}$ is a nonzero ideal, then $\mathfrak{m}_{\mathcal{U}}$ is an int prime ideal.*

Proof. Supposing that $\mathfrak{m}_{\mathcal{U}}$ is a polynomial ideal, we show that $\mathfrak{m}_{\mathcal{U}} = (0)$. For each $\mathfrak{m}_\lambda$, consider the ideal $\mathfrak{M}_{\lambda,0} = \{f \in \text{Int}(D) \mid f(0) \in \mathfrak{m}_\lambda\}$. This is a maximal ideal of Int(D) containing Int$(D, \mathfrak{m}_\lambda)$. It follows from Proposition 1.4 that each $\mathfrak{M}_{\lambda,0}$ is a t-ideal of Int(D). Let $\mathfrak{M}_{\mathcal{U}}$ be the limit of the corresponding family $\{\mathfrak{M}_{\lambda,0}\}$, with respect to the same ultrafilter $\mathcal{U}$. Clearly $\mathfrak{M}_{\mathcal{U}}$ contains X, in particular, it is a nonzero ideal and it follows from Proposition 2.5 that it is a t-ideal of Int(D). Since Int(D) is a PvMD, the localization of Int(D) with respect to $\mathfrak{M}_{\mathcal{U}}$ is a valuation domain. Since $\mathfrak{M}_{\mathcal{U}}$ contains $\mathfrak{m}_{\mathcal{U}}$ and X, and since $\mathfrak{m}_{\mathcal{U}}$ is a polynomial ideal, the localization of Int(D) with respect to $\mathfrak{M}_{\mathcal{U}}$ is the localization of $D[X]$ with respect to $(\mathfrak{m}_{\mathcal{U}}, X)$. Thus $\mathfrak{m}_{\mathcal{U}} = (0)$, since otherwise, this localization is not a valuation domain [Lemma 1.6]. □

We reach our last necessary condition.

Proposition 2.7. *Let D be a domain such that* Int(D) *is a PvMD. Then each nonzero polynomial prime ideal contains a finitely generated ideal which is not contained in any int prime ideal.*

Proof. By way of contradiction, suppose there exists a polynomial prime ideal $\mathfrak{q}$, such that each finitely generated ideal contained in $\mathfrak{q}$ is also contained in some int prime ideal. From Lemma 2.2, it follows that $\mathfrak{q}$ is contained in the limit $\mathfrak{p}_{\mathcal{U}}$ of the family of int prime ideals, with respect to some filter $\mathcal{U}$. It follows from the previous lemma, replacing $\mathcal{U}$, if need be, by an ultrafilter containing it, that we can assume that $\mathfrak{p}_{\mathcal{U}}$ is an int prime ideal. We reach a contradiction, since we can also prove that $\mathfrak{p}_{\mathcal{U}}$ is a polynomial prime ideal. Indeed, $\mathfrak{q} \subseteq \mathfrak{p}_{\mathcal{U}}$. Thus either $\mathfrak{p}_{\mathcal{U}} = \mathfrak{q}$, which is polynomial by hypothesis, or else the height of $\mathfrak{p}_{\mathcal{U}}$ is at least 2, and the conclusion follows from Proposition 1.7. □

If V is a valuation overring of the domain D and $\mathfrak{m}$ is the maximal ideal of V, the *center* of $\mathfrak{m}$ is the intersection $\mathfrak{m} \cap D$ [4, p.218]. If D is a PvMD, then D is integrally closed, thus an intersection of valuation domains $\{V_\lambda\}$. Moreover, each V_λ is essential for D (that is, V_λ is the localization of D with respect to the center of its maximal ideal). Under this condition, we show that a t-maximal ideal in D can be obtained as a limit of such centers.

Proposition 2.8. *Let D be the intersection $D = \bigcap_{\lambda\in\Lambda} V_\lambda$ of a family of valuation domains. For each $\lambda \in \Lambda$, denote by $\mathfrak{p}_\lambda$, the center in D of the maximal ideal of V_λ.*

(i) *If I is a t-ideal of D, then I is contained in the limit $\mathfrak{p}_{\mathcal{U}}$ of the family $\{\mathfrak{p}_\lambda\}$, with respect to some filter $\mathcal{U}$.*

(ii) *If moreover I is maximal, or if I is t-maximal and every V_λ is essential (that is, $V_\lambda = D_{\mathfrak{p}_\lambda}$), then $I = \mathfrak{p}_{\mathcal{U}}$.*

Proof. (i) Let J be an ideal of D. If J is not contained in any $\mathfrak{p}_\lambda$, it is easy to see that that the inverse J^{-1} of J is contained in each V_λ, and hence, that $J_v = J^{-1} = D$. Each finitely generated ideal J contained in I is such that $J_v \subseteq I$, and hence, it is contained in some $\{\mathfrak{p}_\lambda\}$. From Lemma 2.2, it follows that $I \subseteq \mathfrak{p}_{\mathcal{U}}$.
(ii) If moreover I is maximal then obviously $I = \mathfrak{p}_{\mathcal{U}}$. The same conclusion holds if I is t-maximal and if $\mathfrak{p}_{\mathcal{U}}$ is a t-ideal. Now if each V_λ is essential, then each $\mathfrak{p}_\lambda$ is a t-ideal of D [11, Lemma 3.17]. It follows from Proposition 2.5 that $\mathfrak{p}_{\mathcal{U}}$ is then a t-ideal. □

Later on, we consider also limits of rings. We let $\{D_\lambda\}_{\lambda\in\Lambda}$ be a family of rings contained in a field K. For each $x \in K$, we let $C(x)$ be the subfamily of rings containing x (or equivalently, the subset of indices $C(x) = \{\lambda \in \Lambda \mid x \in D_\lambda\}$). Giving ourselves a filter $\mathcal{U}$ over $\{D_\lambda\}$ (or equivalently, over Λ), we let

$$D_{\mathcal{U}} := \{x \in K \mid C(x) \in \mathcal{U}\}.$$

It is easily seen that $D_{\mathcal{U}}$ is a ring. We say that $D_{\mathcal{U}}$ is the *limit of the family $\{D_\lambda\}$, with respect to $\mathcal{U}$*. If each D_λ is an overring of a domain D, with quotient field K, it is clear that the limit $D_{\mathcal{U}}$ is an overring of D.

More particularly, we consider a family $\{v_\lambda\}$ of valuations of a field K, corresponding to a family $\{V_\lambda\}_{\lambda\in\Lambda}$ of valuation domains.

Lemma 2.9. *Let $\{v_\lambda\}_{\lambda\in\Lambda}$ be a family of valuations of a field K, $\mathcal{U}$ be an ultrafilter, and $V_{\mathcal{U}}$ be the limit of the corresponding family $\{V_\lambda\}_{\lambda\in\Lambda}$ of valuation domains, with respect to $\mathcal{U}$. Then $V_{\mathcal{U}}$ is a valuation domain, with quotient field K (or $V_{\mathcal{U}} = K$), and its maximal ideal is $\mathfrak{m}_{\mathcal{U}} := \{x \in K \mid B(x) \in \mathcal{U}\}$, where $B(x) := \{\lambda \mid v_\lambda(x) > 0\}$ (if $V_{\mathcal{U}} = K$, then $\mathfrak{m}_{\mathcal{U}} = (0)$).*

Proof. It is clear that, for each $x \in K$, we have $C(x) = \{\lambda \mid v_\lambda(x) \geq 0\}$. For $x \neq 0$, we have $C(x) \cup C(x^{-1}) = \Lambda$. Since we suppose that $\mathcal{U}$ is an ultrafilter, it follows that $x \in V_{\mathcal{U}}$, or $x^{-1} \in V_{\mathcal{U}}$. Hence $V_{\mathcal{U}}$ is a valuation domain. Now, for each $x \in K$, let $A(x) := \{\lambda \mid v_\lambda(x) = 0\}$. For $x \neq 0$, we have $A(x) = A(x^{-1})$, hence x is a unit of $V_{\mathcal{U}}$ if and only if $A(x) \in \mathcal{U}$. On the other hand, $C(x) = A(x) \cup B(x)$, hence, if $x \in V_{\mathcal{U}}$ (that is, $C(x) \in \mathcal{U}$), but $x \notin \mathfrak{m}_{\mathcal{U}}$ (that is, $B(x) \notin \mathcal{U}$), we have $A(x) \in \mathcal{U}$. It follows that $\mathfrak{m}_{\mathcal{U}}$ is the maximal ideal of $V_{\mathcal{U}}$. □

Finally, the following, relating limits of rings to limits of ideals, follows immediately from the previous lemma.

Proposition 2.10. *Let $\{v_\lambda\}_{\lambda\in\Lambda}$ be a family of valuations of a field K, $\mathcal{U}$ be an ultrafilter on Λ, and D be a domain contained in each corresponding valuation domain V_λ. For each λ, denote by $\mathfrak{p}_\lambda$ the center in D of the maximal ideal of V_λ. Finally, let $V_{\mathcal{U}}$ be the limit of the rings $\{V_\lambda\}$, and $\mathfrak{p}_{\mathcal{U}}$ be the limit of the ideals $\{\mathfrak{p}_\lambda\}$, both with respect to $\mathcal{U}$.*

(i) *Then $\mathfrak{p}_{\mathcal{U}} = D \cap \mathfrak{m}_{\mathcal{U}}$, where $\mathfrak{m}_{\mathcal{U}}$ is the maximal ideal of $V_{\mathcal{U}}$, and $D_{\mathfrak{p}_{\mathcal{U}}} \subseteq V_{\mathcal{U}}$.*

(ii) *If $D_{\mathfrak{p}_{\mathcal{U}}}$ is a valuation domain with quotient field F, then $D_{\mathfrak{p}_{\mathcal{U}}} = V_{\mathcal{U}} \cap F$.*

3. SUFFICIENT CONDITIONS

Lemma 3.1. *Let D be a PvMD. Assume that every int prime ideal of D is an height-one prime ideal. If $\mathfrak{M}$ is a prime ideal of* Int(D) *above an int prime ideal $\mathfrak{m}$, then* Int$(D)_{\mathfrak{M}}$ *is a valuation domain.*

Proof. • Step 1. There is a polynomial $f \in \text{Int}(D)$ with some coefficient x which is not in $D_{\mathfrak{m}}$. Let $\mathcal{P}$ be the family of prime ideals containing the conductor $(D :_D x)$. Each $\mathfrak{p} \in \mathcal{P}$ is an int prime ideal, since $x \notin D_{\mathfrak{p}}$, hence it follows from Corollary 1.3 that $D_{\mathfrak{p}}$ is the ring of a rank-one discrete valuation $v_{\mathfrak{p}}$. The element x satisfies the equivalent conditions of Lemma 1.1. On one hand, there is an integer m such that, for each $\mathfrak{p} \in \mathcal{P}$, if $\beta \notin \mathfrak{p}$, then $(\beta^m - 1) \in \mathfrak{p}$. On the other hand, there is an integer n such that, for each $\mathfrak{p} \in \mathcal{P}$, if $\alpha \in \mathfrak{p}$, then $v_{\mathfrak{p}}(\alpha^n x) \geq 0$. In other words, if we normalize each valuation $v_{\mathfrak{p}}$ in such a way that $v_{\mathfrak{p}}(x) = -1$, there is an integer e such that, for each $\mathfrak{p} \in \mathcal{P}$, and each $\alpha \in D$, $v_{\mathfrak{p}}(\alpha^e)$ is a non-negative integer. In particular, if $\beta \notin \mathfrak{p}$, then $v_{\mathfrak{p}}\big((\beta^m - 1)^e\big)$ is a positive integer.

• Step 2. Let $f \in \text{Int}(D)$. We define a sequence of polynomials, by

$$f_0 = f^e, \text{ and } f_n = f_{n-1}(f_{n-1}^m - 1)^e x = f^e x^n \prod_{i<n} (f_i^m - 1)^e.$$

We claim that, for each n, f_n is an integer-valued polynomial. Letting $a \in D$, and $b_n = f_n(a)$, we show that, for each prime ideal $\mathfrak{q}$ of D, $b_n \in D_{\mathfrak{q}}$. This is immediate if $\mathfrak{q} \notin \mathcal{P}$, that is, $\mathfrak{q}$ does not contain $(D\colon x)$, since $x \in D_{\mathfrak{q}}$ in this case. If $\mathfrak{p} \in \mathcal{P}$, we assume, by induction on n, that $v_{\mathfrak{p}}(b_{n-1})$ is a non-negative integer. If $v_{\mathfrak{p}}(b_{n-1}) > 0$, then $v_{\mathfrak{p}}(b_n) = v_{\mathfrak{p}}(b_{n-1}) - 1 \geq 0$, and if $v_{\mathfrak{p}}(b_{n-1}) = 0$, then $v_{\mathfrak{p}}(b_n) = v_{\mathfrak{p}}\big((b_{n-1}^m - 1)^e x\big)$. Our claim is settled, from Step 1.

• Step 3. We consider an ideal $\mathfrak{M}$ of $\text{Int}(D)$ above $\mathfrak{m}$. If $f_i \in \mathfrak{M}$ for $i < n$, each factor $(f_i^m - 1)^e$ is an element of $\text{Int}(D)$ which is not in $\mathfrak{M}$. Thus either $f_n \in \mathfrak{M}$, and then $f^e x^n \in \mathfrak{M}\,\text{Int}(D)_{\mathfrak{M}}$, or $f_n \notin \mathfrak{M}$, and then $f^e x^n$ is a unit in $\text{Int}(D)_{\mathfrak{M}}$. In conclusion, either $f^e x^n \in \mathfrak{M}\,\text{Int}(D)_{\mathfrak{M}}$ for all n, or, for the smallest integer n such that fails to be true, $f^e x^n$ is a unit in $\text{Int}(D)_{\mathfrak{M}}$.

• Step 4. Let $\varphi \in K(X)$. Write $\varphi = f/g$, where f and g are integer-valued polynomials. For f and g consider sequences (f_n) and (g_n) of polynomials as in Step 2. We may assume that f and g are relatively prime in $K[X]$. Then f^e and g^e are also relatively prime in $K[X]$. Hence, there are polynomials u and v in $D[X]$ and a nonzero $d \in D$ such that $uf^e + vg^e = d$. For each n, we then have $uf^e x^n + vg^e x^n = dx^n$. Since $D_{\mathfrak{m}}$ is completely integrally closed (being a rank-one valuation domain), and since $x \notin D_{\mathfrak{m}}$, there is an integer n such that $dx^n \notin D_{\mathfrak{m}}$. Hence $f^e x^n$ and $g^e x^n$ are not both in $\mathrm{Int}(D)_{\mathfrak{M}}$. A fortiori they are not both in $\mathfrak{M}\,\mathrm{Int}(D)_{\mathfrak{M}}$. From Step 2, if n is the smallest such integer, one of the polynomials $f^e x^n$ and $g^e x^n$ (and may be both of them) is a unit in $\mathrm{Int}(D)_{\mathfrak{M}}$, the other one being anyway an element of the ring $\mathrm{Int}(D)_{\mathfrak{M}}$. Therefore either $(f/g)^e = (f^e x^n)/(g^e x^n)$ or $(g/f)^e = (g^e x^n)/(f^e x^n)$ is in $\mathrm{Int}(D)_{\mathfrak{M}}$. Since D is a PvMD, it is integrally closed, then so is $\mathrm{Int}(D)$ [1, Proposition IV. 4.1]. Thus either φ or φ^{-1} is in $\mathrm{Int}(D)_{\mathfrak{M}}$. In conclusion $\mathrm{Int}(D)_{\mathfrak{M}}$ is a valuation domain. □

Remarks 3.2. (1) This proof is quite similar to that of [1, Proposition VI.4.4], inspired from [13], which gives a sufficient condition for $\mathrm{Int}(D)$ to be a Prüfer domain.
(2) We could prove also a local version of the previous result:
Let D be a PvMD, and $\mathfrak{m}$ be an int prime of D. If $\mathfrak{m}$ is an height-one prime, then each prime ideal $\mathfrak{M}$ of $\mathrm{Int}(D)$ *above $\mathfrak{m}$ is such that* $\mathrm{Int}(D)_{\mathfrak{M}}$ *is a valuation domain.*

We do not assume here that every int prime ideal, and in particular, every prime ideal $\mathfrak{p}$ containing $(D :_D x)$, is an height-one prime. Thus, we can only conclude that $D_{\mathfrak{p}}$ is a valuation domain with finite residue field, whose maximal ideal is principal [Corollary 1.3]. The value group of the corresponding valuation $v_{\mathfrak{p}}$ is then of the form $G \times \mathbb{Z}$, lexicographically ordered, where G is a totally ordered group. Normalizing these valuations in such a way that $v_{\mathfrak{p}}(x) = (0, -1)$, we could nevertheless find, as above, an integer e such that, for each $\mathfrak{p} \in \mathcal{P}$, and each $\alpha \in D$, $v_{\mathfrak{p}}(\alpha^e) = (g, n)$, where $g \in G, g \geq 0$, and if $g = 0$, n is a non-negative integer. The rest of the proof would be the same, using the fact that $\mathfrak{m}$ is an height-one prime, in Step 4, to insure that $D_{\mathfrak{m}}$ be completely integrally closed.

We next consider a polynomial prime ideal $\mathfrak{q}$. The prime ideals of $\mathrm{Int}(D)$ above $\mathfrak{q}$ are in one-to-one correspondence (respecting containment) with the prime ideals of $D_{\mathfrak{q}}[X]$ above $\mathfrak{q}$. The smallest of these ideals is $\mathfrak{q}D_{\mathfrak{q}}[X] \cap \mathrm{Int}(D)$, all the others contain it and are called *uppers to* $\mathfrak{q}$ (they are of the form $\mathfrak{Q} = Q_1 \cap \mathrm{Int}(D)$, where Q_1 is an upper to $\mathfrak{q}$ in $D_{\mathfrak{q}}[X]$).

Lemma 3.3. *Let D be a PvMD and $\mathfrak{q}$ be a nonzero polynomial prime ideal of D. Assume that there exists a finitely generated ideal $I \subseteq \mathfrak{q}$ which is not contained in any int prime ideal of D. Then, the uppers to $\mathfrak{q}$ in* $\mathrm{Int}(D)$ *are not t-ideals.*

Proof. Let $\mathfrak{Q}$ be an upper to $\mathfrak{q}$ in $\mathrm{Int}(D)$ and set $Q := \mathfrak{Q} \cap D[X]$. Then Q is an upper to $\mathfrak{q}$ in $D[X]$. We first claim that $Q_t = D[X]$, that is, Q is not contained in any t-prime ideal of $D[X]$. Indeed, since D is integrally closed, the only t-prime ideals of $D[X]$ are the uppers to zero and the extended primes $\mathfrak{p}[X]$, where $\mathfrak{p}$ is a t-prime ideal of D [9, Lemma 4.5]. If Q were contained in a t-prime ideal $\mathfrak{P}$ of $D[X]$, then, $\mathfrak{P}$ would be of the form $\mathfrak{p}[X]$, for some t-prime ideal $\mathfrak{p}$ of D. But since $D_{\mathfrak{p}}$ is a valuation domain (because D is a PvMD), Q, which is contained in $\mathfrak{P}$, would then also be an extended prime, hence we would have a contradiction. Therefore, there exists a finitely generated ideal $F \subseteq Q$ in $D[X]$, such that $F^{-1} = F_v = D[X]$. By

hypothesis, $\mathfrak{q}$ contains a finitely generated ideal I which is not contained in any int prime ideal of D. Without loss of generality, we can assume that $I \subseteq F$ (replacing F by $F + I$). We set $J := F\operatorname{Int}(D)$. Obviously, we have $J \subseteq \mathfrak{Q}$. We claim that $J_v = (\operatorname{Int}(D) \colon J) = \operatorname{Int}(D)$, so that $\mathfrak{Q}$ is not a t-ideal, finishing the proof. For this, we show that, for each prime ideal $\mathfrak{p}$ of D, we have $(\operatorname{Int}(D) \colon J)_{\mathfrak{p}} = \operatorname{Int}(D)_{\mathfrak{p}}$.
— If $\mathfrak{p}$ is an int prime ideal, since $I \subseteq J$, and $I \not\subseteq \mathfrak{p}$, then $J \not\subseteq \mathfrak{p}$. Thus we have $J\operatorname{Int}(D)_{\mathfrak{p}} = \operatorname{Int}(D)_{\mathfrak{p}}$, and our claim follows.
— If $\mathfrak{p}$ is a polynomial prime ideal, then $\operatorname{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ and $J\operatorname{Int}(D)_{\mathfrak{p}} = F_{\mathfrak{p}}$. From the fact that F is finitely generated and that $(D[X] \colon F) = D[X]$, it follows that $(D_{\mathfrak{p}}[X] \colon F_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$. □

We are ready for our main result. Note that although it is necessary that each nonzero polynomial prime ideal contains a finitely generated ideal which is not contained in any int prime ideal [Proposition 2.7], it is sufficient to restrict this condition to polynomial t-prime ideals.

Theorem 3.4. *Let D be a domain. Then* $\operatorname{Int}(D)$ *is a PvMD if and only if the following conditions hold:*

(a) *D is a PvMD,*
(b) *each int prime ideal of D is an height-one prime ideal,*
(c) *each nonzero polynomial t-prime ideal of D contains a finitely generated ideal which is not contained in any int prime ideal.*

Proof. From Propositions 1.10, 1.7, and 2.7 respectively, these conditions are necessary. Letting $\mathfrak{P}$ be a t-prime ideal of $\operatorname{Int}(D)$, it remains to show that they imply that $\operatorname{Int}(D)_{\mathfrak{P}}$ is a valuation domain. We set $\mathfrak{p} := \mathfrak{P} \cap D$. In fact, if $\mathfrak{p}$ is an int prime ideal, the conclusion follows under conditions (a) and (b) alone, from Lemma 3.1. If $\mathfrak{p}$ is polynomial, we claim that it follows under conditions (a) and (c). Indeed, since $\mathfrak{P}$ is a t-ideal of $\operatorname{Int}(D)$, then $\mathfrak{p}$ is a t-ideal of D [16, Corollary 2.2]. From Lemma 3.3, it follows that $\mathfrak{P}$ is of the form $\mathfrak{P} = \operatorname{Int}(D) \cap \mathfrak{p}D_{\mathfrak{p}}[X]$. Thus $\operatorname{Int}(D)_{\mathfrak{P}}$ is also the localization of $D_{\mathfrak{p}}[X]$ with respect to $\mathfrak{p}D_{\mathfrak{p}}[X]$. Since $\mathfrak{p}$ is a t-ideal, $D_{\mathfrak{p}}$ is the ring of a valuation v. The localization of $D_{\mathfrak{p}}[X]$ with respect to $\mathfrak{p}D_{\mathfrak{p}}[X]$ is then the ring of the valuation extending v to $K(X)$, defined on a polynomial f by taking the infimum of the valuations of its coefficients. □

4. CANONICAL INTERSECTION

We partition the spectrum $\operatorname{Spec}(D)$ of a domain D into two subsets Δ_0 and Δ_1, where

- Δ_0 is the set of int prime ideals,
- Δ_1 is the set of polynomial prime ideals.

Recall that each prime in Δ_0 is maximal. We then set

$$D_0 := \bigcap_{\mathfrak{m} \in \Delta_0} D_{\mathfrak{m}}, \quad D_1 := \bigcap_{\mathfrak{p} \in \Delta_1} D_{\mathfrak{p}}.$$

Obviously $D = D_0 \cap D_1$. With these notations, we also have the following.

Lemma 4.1. *Let D be a domain, then*

$$\operatorname{Int}(D) = \operatorname{Int}(D_0) \cap \operatorname{Int}(D_1) = \operatorname{Int}(D_0) \cap D_1[X].$$

Proof. For each prime $\mathfrak{p}$ we have the containments

$$\mathrm{Int}(D) \subseteq \mathrm{Int}(D, D_{\mathfrak{p}}) = \mathrm{Int}(D_{\mathfrak{p}}).$$

Hence,

$$\mathrm{Int}(D) = \bigcap_{\mathfrak{p}\in \mathrm{Spec}(D)} \mathrm{Int}(D_{\mathfrak{p}}) = \Big(\bigcap_{\mathfrak{p}\in\Delta_0} \mathrm{Int}(D_{\mathfrak{p}})\Big) \bigcap \Big(\bigcap_{\mathfrak{p}\in\Delta_1} \mathrm{Int}(D_{\mathfrak{p}})\Big),$$

The first equality follows. On the other hand, $\mathrm{Int}(D) \subseteq D_{\mathfrak{p}}[X]$ for each $\mathfrak{p} \in \Delta_1$, and thus

$$\mathrm{Int}(D) \subseteq \bigcap_{\mathfrak{p}\in\Delta_1} D_{\mathfrak{p}}[X] = D_1[X] \subseteq \mathrm{Int}(D_1).$$

The second equality follows. □

Remarks 4.2. (1) In general, if D is the intersection of two overrings: $D = B_0 \cap B_1$, it does not necessarily follow that $\mathrm{Int}(D) = \mathrm{Int}(B_0) \cap \mathrm{Int}(B_1)$ (in fact, if B is an overring of D, $\mathrm{Int}(B)$ does not necessarily contain $\mathrm{Int}(D)$, as for instance, if B is the integral closure of D [1, Exercises IV.28 and IV.29]). The proof above shows that the equality $\mathrm{Int}(D) = \mathrm{Int}(B_0) \cap \mathrm{Int}(B_1)$ holds when each overring is an intersection of localizations of D.
(2) There are examples such that $\mathrm{Int}(D_1) \neq D_1[X]$. For instance, Example VI.4.17 of [1] provides an almost Dedekind domain D with finite residue fields and a single maximal ideal $\mathfrak{m}$ which is polynomial. In this example, we thus have $D_1 = D_{\mathfrak{m}}$, however $D_{\mathfrak{m}}[X] \neq \mathrm{Int}(D_{\mathfrak{m}})$ since $D_{\mathfrak{m}}$ is a rank-one discrete valuation domain with finite residue field.
(3) If Θ is a set of prime ideals of D, such that $D = \bigcap_{\mathfrak{p}\in\Theta} D_{\mathfrak{p}}$, we could restrict ourselves to its subsets $\Theta_0 = \Delta_0 \cap \Theta$ and $\Theta_1 = \Delta_1 \cap \Theta$ to reach the same conclusions. Letting $D_{0,\Theta} := \bigcap_{\mathfrak{m}\in\Theta_0} D_{\mathfrak{m}}$, and $D_{1,\Theta} := \bigcap_{\mathfrak{p}\in\Theta_1} D_{\mathfrak{p}}$, we would obtain, as above, the equality $\mathrm{Int}(D) = \mathrm{Int}(D_{0,\Theta}) \cap D_{1,\Theta}[X]$. As in [16], we could for instance let Θ be the set of t-prime ideals of D (note that, in this case $\Theta_0 = \Delta_0$, since each int prime ideal is a t-ideal [Proposition 1.2]).

When $\mathrm{Int}(D)$ is a PvMD, each int prime ideal $\mathfrak{m}$ of D is such that $D_{\mathfrak{m}}$ is the ring of a rank-one discrete valuation with finite residue field [Corollary 1.8]. We shall prove that this family of valuations satisfies some double-boundedness condition and derive that $\mathrm{Int}(D_0)$ is a Prüfer domain. But first, we set a definition, inspired by [14].

Definition 4.3. Let $\{V_\lambda\}_{\lambda\in\Lambda}$ be a family of rank-one discrete valuation domains of a field K. Denote by v_λ each corresponding normalized valuation and by $\mathfrak{m}_\lambda$ the maximal ideal of V_λ. We say that $\{v_\lambda\}_{\lambda\in\Lambda}$ (or $\{V_\lambda\}_{\lambda\in\Lambda}$) *satisfies the double boundedness condition* if, for each nonzero $x \in K$, there is an integer n such that $v_\lambda(x) \leq n$, and $|V_\lambda/\mathfrak{m}_\lambda| \leq n$, for each $\lambda \in \Lambda$ such that $v_\lambda(x) > 0$.

Remarks 4.4. (1) The definition implies that each valuation domain V_λ has a finite residue field.
(2) In [14] the double boundedness condition, is restricted to the nonzero elements of a domain D of which each V_λ is an overring. This may appear to be weaker, but in fact, if $x \in K$, then $x = a/b$, with a, b in D. Hence, $v_\lambda(x) \leq v_\lambda(a)$ for each λ (since $v_\lambda(b) \geq 0$) and in particular, $v_\lambda(x) > 0$ implies $v_\lambda(a) > 0$.

Next we consider the intersection of a family of rank-one discrete valuation domains which satisfies the double boundedness condition; we show that it is an almost Dedekind domain with finite residue fields. But first, we establish a lemma on the limit of such a family, with respect to an ultrafilter.

Lemma 4.5. *Let $\{v_\lambda\}_{\lambda\in\Lambda}$ be a family of rank-one discrete valuation domains of a field K and $\mathcal{U}$ be an ultrafilter. If $\{v_\lambda\}$ satisfies the double boundedness condition, then the limit $V_{\mathcal{U}}$ of the corresponding family $\{V_\lambda\}_{\lambda\in\Lambda}$ of valuation domains, with respect to $\mathcal{U}$, is the field K or a rank-one discrete valuation domain, with finite residue field.*

Proof. We denote by V_λ the ring of the valuation v_λ. From Lemma 2.9, we already know that $V_{\mathcal{U}}$ is a valuation domain, with quotient field K, and with maximal ideal $\mathfrak{m}_{\mathcal{U}} := \{x \in K \mid B(x) \in \mathcal{U}\}$, where $B(x) := \{\lambda \mid v_\lambda(x) > 0\}$. It may happen that $V_{\mathcal{U}} = K$, but if this is not the case, it remains to prove that the corresponding valuation $v_{\mathcal{U}}$ is rank-one discrete, and that its residue field is finite.
— Consider two nonzero elements a and b in $\mathfrak{m}_{\mathcal{U}}$. Since $v_\lambda(a)$ is bounded, there exists an integer e such that, $v_\lambda(b) > 0$, implies $ev_\lambda(b) > v_\lambda(a)$. In particular, for $\lambda \in B(a) \cap B(b)$, we have $v_\lambda(b^e/a) > 0$. On the other hand, $\big(B(a) \cap B(b)\big) \in \mathcal{U}$, hence $(b^e/a) \in \mathfrak{m}_{\mathcal{U}}$. Since there is such an integer e for each a and b in $\mathfrak{m}_{\mathcal{U}}$, the valuation $v_{\mathcal{U}}$ is rank-one discrete.
— Choose arbitrarily a nonzero in $\mathfrak{m}_{\mathcal{U}}$. Since $|V_\lambda/\mathfrak{m}_\lambda|$ is bounded for $\lambda \in B(a)$, there exists an integer q such that, for each $z \in K$, and each $\lambda \in B(a)$, $v_\lambda(z) \geq 0$ implies $v_\lambda(z^q - z) > 0$, that is, $B(z^q - z)$ contains the intersection $C(z) \cap B(a)$, where $C(z) = \{\lambda \in \Lambda \mid v_\lambda(z) \geq 0\}$. If $z \in V_{\mathcal{U}}$, then $C(z) \in \mathcal{U}$, hence $C(z) \cap B(a) \in \mathcal{U}$. A fortiori, $B(z^q - z) \in \mathcal{U}$. Therefore, for each $z \in V_{\mathcal{U}}$, we have $(z^q - z) \in \mathfrak{m}_{\mathcal{U}}$, hence the residue field of $V_{\mathcal{U}}$ has at most q elements. □

Proposition 4.6. *Let $D = \bigcap_{\lambda\in\Lambda} V_\lambda$ be the intersection of a family $\{V_\lambda\}_{\lambda\in\Lambda}$ of rank-one discrete valuation domains with quotient field K. If $\{V_\lambda\}$ satisfies the double boundedness condition, then D is a field or an almost Dedekind domain with finite residue fields.*

Proof. We denote by v_λ the valuation corresponding to V_λ. If D is not a field, we consider a nonzero maximal ideal $\mathfrak{m}$ of D and first prove that $D_\mathfrak{m}$ is a valuation domain. If a and b are two nonzero elements in D, we show that either (a/b) or (b/a) belongs to $D_\mathfrak{m}$, and our four steps argument bears some similarity with the proof of Lemma 3.1. Finally we show that, in fact, $D_\mathfrak{m}$ is a rank-one discrete valuation domain with finite residue field.

- Step 1. Set $B(a) := \{\lambda \mid v_\lambda(a) > 0\}$. Since $|V_\lambda/\mathfrak{m}_\lambda|$ is bounded for $\lambda \in B(a)$, there exists an integer m such that, for $\lambda \in B(a)$, $v_\lambda(z) = 0$ implies $v_\lambda(z^m - 1) > 0$. Since $v_\lambda(a)$ is bounded, there exists an integer e such that, normalizing each valuation $\lambda \in B(a)$ in such a way that $v_\lambda(z) = 1$, for each $\lambda \in B(a)$ and $z \in D$, $v_\lambda(z^e)$ is a non-negative integer.
- Step 2. Consider the sequence $\{b_n\}$ defined by

$$b_0 = b^e, \ldots, b_n = \frac{b_{n-1}}{a}(b_{n-1}^m - 1)^e = \frac{b^e}{a^n}\prod_{i<n}(b_i^m - 1)^e.$$

By induction, it follows from the definition of the integers e and m that $v_\lambda(b_n)$ is a non-negative integer, for each n and each $\lambda \in B(a)$. Therefore $b_n \in D$, for each n (since on the other hand, $v_\lambda(a) = 0$, for each $\lambda \notin B(a)$).

• Step 3. If $b \notin \mathfrak{m}$, then obviously $(a/b) \in D_{\mathfrak{m}}$, and we are done. Hence we suppose that $b_i \in \mathfrak{m}$, for $i < n$. Thus $\prod_{i<n}(b_i^m - 1)^e \notin \mathfrak{m}$. Writing

$$\frac{b^e}{a^n} = \frac{b_n}{\prod_{i<n}(b_i^m - 1)^e},$$

it follows that $(b^e/a^n) \in D_{\mathfrak{m}}$ and that $(b^e/a^n) \in \mathfrak{m}D_{\mathfrak{m}}$ if and only if $b_n \in \mathfrak{m}$. We then consider two cases, showing that either $(b/a)^e \in D_{\mathfrak{m}}$, or $(a/b)^e \in D_{\mathfrak{m}}$:
— $b_i \in \mathfrak{m}$, for each $i < e$. Then $(b^e/a^e) \in D_{\mathfrak{m}}$.
— $b_i \notin \mathfrak{m}$, for some $i < e$. Let m be the smallest such integer. We have discarded the case $m = 0$, thus $1 \le m \le (e-1)$. Since $b_i \in \mathfrak{m}$, for $i < m$, we have $(b^e/a^m) \in D_{\mathfrak{m}}$, and since $b_m \notin \mathfrak{m}$, we have $(b^e/a^m) \notin \mathfrak{m}D_{\mathfrak{m}}$. Thus (b^e/a^m) is a unit in $D_{\mathfrak{m}}$, whence $(a^m/b^e) \in D_{\mathfrak{m}}$. Multiplying by a^{e-m}, we obtain $(a^e/b^e) \in D_{\mathfrak{m}}$.
• Step 4. Since D is the intersection of valuation domains, D is integrally closed, hence so is $D_{\mathfrak{m}}$. Since either $(b/a)^e \in D_{\mathfrak{m}}$, or $(a/b)^e \in D_{\mathfrak{m}}$, it follows that either $(b/a) \in D_{\mathfrak{m}}$, or $(a/b) \in D_{\mathfrak{m}}$, and we conclude that $D_{\mathfrak{m}}$ is a valuation domain.
• Step 5. From the first four steps, D is a Prüfer domain. Thus every ideal of D is a t-ideal. Consider a maximal ideal $\mathfrak{m}$. Denoting by $\mathfrak{p}_\lambda$ the center of the maximal ideal of V_λ in D. It follows from Proposition 2.8 that $\mathfrak{m} = \mathfrak{p}_{\mathcal{U}}$, where $\mathfrak{p}_{\mathcal{U}}$ is the limit of the family $\{\mathfrak{p}_\lambda\}$, with respect to some ultrafilter $\mathcal{U}$. Since $D_{\mathfrak{m}}$ is a valuation domain, it follows from Proposition 2.10 that $D_{\mathfrak{m}} = V_{\mathcal{U}} \cap F$, where $V_{\mathcal{U}}$ is the limit of the family $\{V_\lambda\}$, with respect to $\mathcal{U}$, and where F is the quotient field of D (contained in K). Finally, it follows from Lemma 4.5 that $D_{\mathfrak{m}}$ is the ring of a rank-one discrete valuation domain with finite residue field. □

In this situation, we finally conclude that Int(D) is a Prüfer domain.

Proposition 4.7. *Let* $D = \bigcap_{\lambda\in\Lambda} V_\lambda$ *be the intersection of a family* $\{V_\lambda\}_{\lambda\in\Lambda}$ *of rank-one discrete valuation domains with quotient field* K. *If* $\{V_\lambda\}$ *satisfies the double boundedness condition, then* Int(D) *is a Prüfer domain.*

Proof. If D is a field, then Int$(D) = D[X]$ is a principal ideal domain. Otherwise, let $\mathfrak{m}$ be a (nonzero) maximal ideal of D. It is enough to show that $\mathfrak{m}$ is an int prime. Indeed, since D is a one-dimensional PvMD, it then follows from Lemma 3.1 that, for each prime $\mathfrak{M}$ of Int(D) above $\mathfrak{m}$, Int$(D)_{\mathfrak{M}}$ is a valuation domain. We choose a nonzero element $y \in \mathfrak{m}$, and let $x = y^{-1}$. We set $B(y) := \{\lambda \mid v_\lambda(y) > 0\}$. From the double boundedness condition, we derive that there are integers e and q such that, for each $a \in D$ and each $\lambda \in B(y)$, we have $v_\lambda\big((a^q - a)^e\big) \ge v_\lambda(y)$. It follows that the polynomial $x(X^q - X)^e$ is integer-valued. The conclusion follows since $x \notin D_{\mathfrak{m}}$. □

If Int(D) is a PvMD, then for each int prime ideal $\mathfrak{m}$ of D, $D_{\mathfrak{m}}$ is a discrete rank-one valuation domain [Corollary 1.8]. We show that the corresponding family of valuations satisfies the double boundedness condition.

Proposition 4.8. *Let* D *be a domain such that* Int(D) *is a PvMD. The family* $\{v_{\mathfrak{m}}\}_{\mathfrak{m}\in\Delta_0}$ *of valuations corresponding to the int primes of* D *satisfies the double boundedness condition.*

Proof. Suppose by way of contradiction that, for some $x \in D$, either $v_{\mathfrak{m}}(x)$ or $|D/\mathfrak{m}|$ is not bounded: we may consider a sequence $\{v_n\}$ of valuations in Δ_0, such that, for each n, $v_n(x) > 0$ and either $v_n(x) \ge n$, or $|D/\mathfrak{m}_n| \ge n$ (denoting by $\mathfrak{m}_n$ the int prime ideal corresponding to v_n). We consider an ultrafilter $\mathcal{U}$ over the

integers, containing the cofinite sets, and we let $\mathfrak{m}_{\mathcal{U}}$ be the limit prime ideal of the family $\{\mathfrak{m}_n\}$, with respect to $\mathcal{U}$. This ideal contains x, and thus, is a nonzero ideal. From Proposition 2.6, $\mathfrak{m}_{\mathcal{U}}$ is an int prime ideal and in particular, $D_{\mathfrak{m}_{\mathcal{U}}}$ is the ring of a rank-one discrete valuation $v_{\mathcal{U}}$. Analogously to a recent paper [14], we claim however that $\mathrm{Int}(D) \subseteq D_{\mathfrak{m}_{\mathcal{U}}}[X]$, reaching a contradiction. Letting $f \in \mathrm{Int}(D)$, and a be one of its coefficients, we show that $v_{\mathcal{U}}(a) \geq 0$.
— Consider the case where $|D/\mathfrak{m}_n| \geq n$. It follows from [1, Corollary I.3.3], that $v_n(a) \geq 0$, if $|D/\mathfrak{m}_n| > \deg(f) =: d$. Hence $v_n(a) \geq 0$, for $n > d$. Letting $C(a) := \{n \in \mathbb{N} \mid v_n(a) \geq 0\}$, we thus have $C(a) \in \mathcal{U}$, and our claim follows from Proposition 2.10.
— Consider the case where $v_n(x) \geq n$. Assume, by way of contradiction, that $v_{\mathcal{U}}(a) < 0$. Since $x \in \mathfrak{m}_{\mathcal{U}}$, it follows that, for some positive integers p and q, $v_{\mathcal{U}}(a^p x^q) = 0$. Hence there is some set $U \in \mathcal{U}$, such that, for $n \in U$, we have $v_n(a^p x^q) = 0$, and thus $v_n(a) = -(q/p)v_n(x) \leq -(q/p)n$. On the other hand, it follows from [1, Corollary II.2.13] that we have $v_n(a) \geq -\deg(f)$. Since U is infinite, there is $n \in U$, such that $n > (p/q)\deg(f)$. We reach a contradiction. □

From Propositions 4.7 and 4.8, we reach the following conclusion.

Corollary 4.9. *Let D be a domain such that* $\mathrm{Int}(D)$ *is a PvMD. Then the overring $D_0 = \bigcap_{\mathfrak{m}\in\Delta_0} D_{\mathfrak{m}}$ of D is such that* $\mathrm{Int}(D_0)$ *is a Prüfer domain.*

5. COUNTEREXAMPLES

In this section, we use a construction often considered to provide almost Dedekind domains showing various properties, as in [1, pp.148–151] or [5]. The general setting in the following examples is the following.

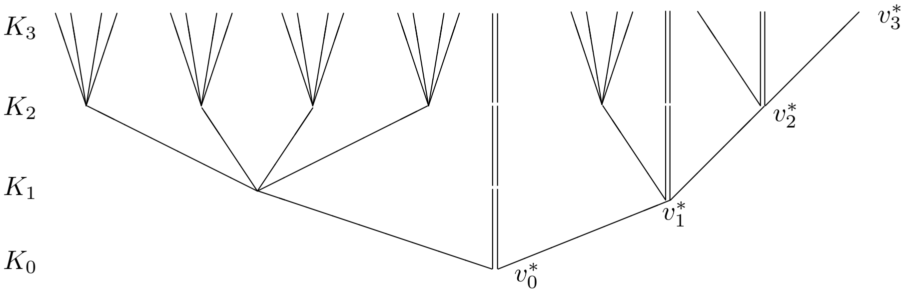
Let $\{K_n\}_{n\geq 0}$ be an ascending sequence of finite algebraic extensions of $\mathbb{Q}$, with $K_0 := \mathbb{Q}$, and set $K := \bigcup_n K_n$. Fix a prime number p and let $V_0 := \mathbb{Z}_{(p)}$. Let D_n be the integral closure of V_0 in K_n (which is a semilocal Dedekind domain) and D the integral closure of V_0 in K. Then $D = \bigcup_n D_n$. Set v_0^* the valuation associated to V_0 (the p-adic valuation). Using Hasse's existence theorem [7], we can prescribe the way v_0^* decomposes from K_0 to K_1, and then, by induction on n, we can also prescribe the way each valuation previously constructed in K_{n-1} decomposes in K_n. For instance, in these examples, we do not allow any ramification. Thus D is an almost Dedekind domain (and thus, a PvMD), the valuations in K extending v_0^* are discrete and they are the essential valuations of D.

Example 5.1. *An almost Dedekind domain D such that* $\mathrm{Int}(D_0)$ *is a Prüfer domain but such that* $\mathrm{Int}(D)$ *is not a PvMD.*

We let each K_n be an extension of degree four of K_{n-1}, and prescribe the decomposition of each valuation — with no ramification — in a such a way that, for each prime ideal $\mathfrak{p}$ of D, either $D/\mathfrak{p} \cong \mathbb{Z}/(p)$ or $D/\mathfrak{p}$ is infinite. This assures the double-boundedness condition for the int prime ideals. One key aspect of the construction is that it gives a maximal ideal p^* of D such that $D/\mathfrak{p}^* \cong \mathbb{Z}/(p)$, and such that, for each positive integer n, there exists another prime ideal $\mathfrak{p}_n$, with $D/\mathfrak{p}_n$ infinite such that $\mathfrak{p}_n \cap D_n = \mathfrak{p}^* \cap D_n$. The other key aspect of the construction is that it gives the existence, for each n, of an int prime ideal $\mathfrak{q}_n$ of D such that $\mathfrak{q}_n \cap D_n = \mathfrak{p}^* \cap D_n$. We proceed as follows:

— From K_0 to K_1, we extend v_0^* in three valuations, u_1, v_1, and v_1^*, in such a way that the residual degree of the extension v_1/v_0^* is $f = 2$, while u_1 and v_1^* are immediate extensions of v_0^* (their residual degree is $f = 1$).
— From K_1 to K_2, we extend v_1, in a single extension with a residual degree $f = 4$, we totally decompose u_1, in four immediate extensions, and we decompose v_1^* in three extensions, in exactly the same way as we did for v_0^*, one with $f = 2$, and two immediate extensions.
— By induction hypothesis, some valuations in K_{n-1} are immediate extensions of v_0^*, the others extending v_0^* with a residual degree $f \geq 2$. A single triplet extends v_{n-2}^*, one with $f = 2$, and two immediate extensions of v_0^* (hence a fortiori of v_{n-2}^*). We denote one of the immediate extensions of v_{n-2}^* by v_{n-1}^*.
— From K_{n-1} to K_n, we decompose v_{n-1}^* in three extensions, as we did for v_0^*, one with $f = 2$, two with $f = 1$, we totally decompose each other immediate extension of v_0^* in four extensions, and finally we extend all non-immediate extension of v_0^* in a single extension, with $f = 4$.

We illustrate this construction by a tree. A single line represents an immediate extension, and a multiple line an extension with $f \geq 2$.



We denote by v^* the single valuation of K which extends each v_i^*, by $\{u_\lambda\}_{\lambda\in\Lambda}$ the family of all other immediate extensions of v_0^* in K, and finally we denote by $\{v_\gamma\}_{\gamma\in\Gamma}$ the family of non-immediate extensions of v_0^* in K. The residue field of v^* and of each u_λ is finite (with p elements); the residue field of each v_γ is infinite.

The almost Dedekind domain D is the intersection of all the corresponding valuation domains. We show that it is the desired counterexample.

- *The prime ideals of D corresponding to v^* and to each v_γ are polynomial primes.* Since the residue field of each v_γ is infinite, it is clear that the corresponding primes are polynomial. Let now $\mathfrak{p}^*$ be the prime ideal corresponding to v^*. Its residue field is finite, but we claim that it is a polynomial prime, that is $\mathrm{Int}(D) \subseteq D_{\mathfrak{p}^*}[X]$. Let $f \in \mathrm{Int}(D)$. There is an integer r such that $f \in K_r[X]$. The restriction of v^* to K_r coincides with the restriction of some valuation v_γ. Since v_γ corresponds to a polynomial prime $\mathfrak{p}_\gamma$, we have $f \in D_{\mathfrak{p}_\gamma}[X]$, and hence, $f \in D_{\mathfrak{p}^*}[X]$.
- *The prime ideals of D corresponding to each u_λ are int prime ideals.* Consider a valuation u_{λ_0} in this family. We claim there exists an element $t \in D$ such that $u_{\lambda_0}(t) = 1$ and $(X^p - X)/t$ is an integer-valued polynomial. First, there is an integer r such that u_{λ_0}, and v^* do not have the same restriction to K_r. In particular, each essential valuation of D having the same restriction as u_{λ_0} to K_r belongs to the family $\{u_\lambda\}_{\lambda\in\Lambda}$, hence its residue field has p elements. By the approximation theorem in the Dedekind domain $D_r = D \cap K_r$, there is an element t such that $u_{\lambda_0}(t) = 1$, while the valuation of t is 0, for all other essential valuation of D_r. In

other words, for each essential valuation of D, the valuation of t is either 0 or 1, and if it is 1, its residue field has p elements. Our claim is settled.

- $\mathrm{Int}(D_0)$ *is a Prüfer domain.* We can conclude, from the previous statements, that $\Delta_0 = \{u_\lambda\}_{\lambda\in\Lambda}$. Since each u_λ is an immediate extension of the p-adic valuation in $\mathbb{Q}$, Δ_0 satisfies the double-boundedness condition. It follows from Proposition 4.7 that $\mathrm{Int}(D_0)$ is a Prüfer domain.
- *If* $I := (x_1, \cdots, x_n)$ *is a finitely generated ideal contained in the prime ideal* $\mathfrak{p}^*$ (*corresponding to* v^*), *then* I *is contained in an int prime ideal* $\mathfrak{p}_\lambda$ (*corresponding to some* u_λ). Since I is contained in $\mathfrak{p}^*$, we have $v^*(x_j) > 0$, for $1 \leq j \leq n$. There exists an integer r such that K_r contains x_j, for $1 \leq j \leq n$. The restriction of some valuation u_λ coincides with v^* on K_r. It follows that $u_\lambda(x_j) > 0$, for $1 \leq j \leq n$. Hence I is contained in the corresponding prime $\mathfrak{p}_\lambda$.

Remarks 5.2. (1) We see that the condition "$\mathrm{Int}(D_0)$ is a Prüfer domain" is not sufficient for $\mathrm{Int}(D)$ to be a PvMD. However, it implies condition (b) of Theorem 3.4. Indeed it implies that D_0 is an almost Dedekind domain (with finite residue fields), and hence, that each int prime ideal of D is an height-one prime.

(2) Recall that a Krull-type domain is a locally finite intersection $D = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_\mathfrak{p}$, where $\mathcal{P}$ is a family of prime ideals and $D_\mathfrak{p}$ is a valuation domain for each $\mathfrak{p} \in \mathcal{P}$. For example, *Krull domains* or *generalized Krull domains* are Krull-type domains [4, p. 524]. The condition that $\mathrm{Int}(D_0)$ is a Prüfer domain is enough for $\mathrm{Int}(D)$ to be a PvMD in this case. First, a Krull-type domain is itself a PvMD, thus satisfies condition (a) of Theorem 3.4. It follows from the previous remark that condition (b) is satisfied. Finally we claim that we have condition (c). Indeed, one can also write $D = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_\mathfrak{p}$, where $\mathcal{P}$ is the set of t-prime ideals of D, this intersection being locally finite [6]. Hence each ideal $\mathfrak{p}$ of $\mathcal{P}$ contains a finitely generated ideal I which is not contained in any other ideal $\mathfrak{p}$ of $\mathcal{P}$ (that is, which is not contained in any other t-prime ideal). We recover [16, Theorem 3.2].

(3) Under the condition that D is a PvMD and that $\mathrm{Int}(D_0)$ is a Prüfer domain, $\mathrm{Int}(D)$ is the intersection of two PvMDs. Indeed, it follows from Remark 4.2 (3) that we have $\mathrm{Int}(D) = \mathrm{Int}(D_0) \cap D_{1,\Theta}[X]$, with $D_{1,\Theta} := \bigcap_{\mathfrak{p}\in\Theta_1} D_\mathfrak{p}$, where Θ_1 is the set of polynomial t-prime ideals. If D is a PvMD, then so is $D_{1,\Theta}$, as it is is an intersection of localizations at t-prime ideals, and thus, so is $D_{1,\Theta}[X]$. The previous example shows that this intersection is not in general itself a PvMD. In that case also, $\mathrm{Int}(D)$ is essential, that is, $\mathrm{Int}(D)$ is an intersection of localizations which are valuation domains (since this holds for both $\mathrm{Int}(D_0)$ and $D_{1,\Theta}[X]$), yet is not a PvMD.

We note that this example gives a natural answer to the question, raised by M. Griffin in [6], whether an essential domain should be a PvMD. W. Heinzer and J. Ohm have already constructed a counterexample to this conjecture in [8]. The ring of integer-valued polynomials that we have constructed above has the advantage of being more natural and simpler than the example provided in [8].

Example 5.3. *An almost Dedekind domain* D *such that* $\mathrm{Int}(D)$ *is a PvMD with a polynomial prime* $\mathfrak{p}$ *such that* $\mathrm{Int}(D_\mathfrak{p}) \neq D_\mathfrak{p}[X]$.

To obtain the desired example, we let here each K_n be an extension of degree three of K_{n-1}, and we prescribe the decomposition of each valuation as follows:
— From K_0 to K_1, we extend v_0^* in two valuations v_1 and v_1^*, v_1 with residual degree $f = 2$ and v_1^* an immediate extension of v_0^*.

— By induction hypothesis, there are $n-1$ extensions of v_0^* in K_{n-1}, only one of them, denoted by v_{n-1}^* is an immediate extension, the others having a residual degree $f \geq 2$.
— From K_{n-1} to K_n we decompose v_{n-1}^* in two extensions, as we did for v_0^*, one with $f = 2$ and one with $f = 1$. We extend all the other (non-immediate) extensions of v_0^* in a single extension, with $f = 3$.

We denote by v^* the single valuation of K which extends each v_i^*. Then v^* is an immediate extension of v_0^*. All the other extensions, forming a family $\{v_\gamma\}_{\gamma \in \Gamma}$, have an infinite residue field.

The almost Dedekind domain D is the intersection of the corresponding valuation domains V^* and $\{V_\gamma\}_{\gamma \in \Gamma}$. We show that it is the desired counterexample.
• $\mathrm{Int}(D)$ *is a PvMD.* In fact, we show that $\mathrm{Int}(D) = D[X]$. Let $f \in \mathrm{Int}(D)$. Since each v_γ has an infinite residue field, it is clear that $f \in V_\gamma[X]$. On the other hand, there is an integer r such that $f \in K_r[X]$. The restriction of v^* to K_r coincides with the restriction of some valuation v_γ. Thus we have also $f \in V^*[X]$.
• *The prime* $\mathfrak{p}^*$ *corresponding to* v^* *is a polynomial prime, but* $\mathrm{Int}(D_{\mathfrak{p}^*}) \neq D_{\mathfrak{p}^*}[X]$. Since $\mathrm{Int}(D) = D[X]$, every prime ideal is obviously polynomial. On the other hand $D_{\mathfrak{p}^*}$ is a rank-one discrete valuation domain with finite residue field, hence the second assertion follows from [1, Proposition I.3.16].

References

[1] P.-J. Cahen and J.-L. Chabert, *Integer-Valued Polynomials*, Amer. Math. Soc. Surveys and Monographs, **48**, Providence, 1997.

[2] P.-J. Cahen, J.-L. Chabert, and S. Frisch, *Interpolation domains*, to appear.

[3] J.-L. Chabert, *Integer-Valued Polynomials, Prüfer Domains and Localization*, Proc. Amer. Math. Soc. **118** (1993), 1061–1073.

[4] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972; rep. Queen's Papers in Pure and Applied Mathematics, vol. 90, Queen's University, Kingston, 1992.

[5] R. Gilmer, *Prüfer domains and rings of integer-valued polynomials*, J. Algebra **129** (1990), 502–517.

[6] M. Griffin, *Some Results on v-multiplication Rings*, Canad. J. Math. **10** (1967), 710–722.

[7] H. Hasse, *Zwei Existenztheoreme über algebraische Zahlkörper*, Math. Ann. **95** (1925), 229–238.

[8] W. Heinzer - J. Ohm, *An essential ring which is not a v-multiplication ring*, Can. J. Math. **25** (1973), 856–861.

[9] E.G. Houston - M. Zafrullah, *Integral Domains in which each t-ideal is divisorial*, Michigan Math. J. **35** (1988), 291–300.

[10] P. Jaffard *Les systèmes d'ideaux* Dunod, Paris, 1960.

[11] B. G. Kang, *Prüfer v-Multiplication Domains and the Ring* $R[X]_{N_v}$, J. Algebra **123** (1989), 151–170.

[12] B. G. Kang, *Some questions about Prüfer v-Multiplication Domains*, Comm. Algebra **17(3)** (1989), 553–564.

[13] A. Loper, *Sequence domains and integer-valued polynomials*, J. Pure Appl. Algebra **119** (1997), 185–210.

[14] ———, *A classification of all domains D such that* $\mathrm{Int}(D)$ *is a Prüfer domain*, to appear in Proc. Amer. Math. Soc.

[15] D. E. Rush, *The conditions* $\mathrm{Int}(R) \subseteq R_S[X]$ *and* $\mathrm{Int}(R_S) = \mathrm{Int}(R)_S$ *for integer-valued polynomials*, J. Pure Appl. Algebra **125** (1998), 287–303.

[16] F. Tartarone, *Integer-Valued Polynomials over Krull-type Domains and Prüfer v-Multiplication Domains*, to appear in Proc. Amer. Math. Soc.

[17] M. Zafrullah, *The v-operation and intersection of quotient rings of integral domains*, Comm. Algebra, **13 (8)** (1985), 1699–1712.

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