SKOLEM PROPERTIES AND INTEGER-VALUED POLYNOMIALS (A SURVEY)

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Abstract. Let $\mathrm{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}$ be the ring of integer-valued polynomials on a domain D with quotient field K. One says that $\mathrm{Int}(D)$ has the strong Skolem property if the finitely generated ideals of $\mathrm{Int}(D)$ are characterized by their values, that is, $\mathfrak{A} = \mathfrak{B}$ if and only if, for each $a \in D$, $\mathfrak{A}(a) = \{g(a) \mid g \in \mathfrak{A}\}$ is equal to $\mathfrak{B}(a) = \{g(a) \mid g \in \mathfrak{B}\}$. For example, it is well known that, if D is the ring of integers of a number field, then $\mathrm{Int}(D)$ has the strong Skolem property.

After a survey of the main known results, we show how these results may be extended to the ring $\mathrm{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}$ of integer-valued polynomials on a subset E of D. In particular, if D is Noetherian, local, one-dimensional, and analytically irreducible, we show that the finitely generated ideals of $\mathrm{Int}(E, D)$ containing nonzero constants are characterized by their values if and only if the topological closure $\widehat{E}$ of E (in the topology defined by the maximal ideal) is compact.

Introduction

Let E be a set, D be a ring, $\mathcal{F}(E, D)$ be the ring of functions from E to D, and R be a subring of $\mathcal{F}(E, D)$. For instance, R may be the ring $\mathcal{C}(E, D)$ of continuous functions, if E and D are topological spaces. If D is a domain, we shall mainly be concerned by the ring

$$\mathrm{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}$$

of integer-valued polynomials on some subset E of the quotient field K of D. More particularly, for $E = D$, we denote by $\mathrm{Int}(D)$ the ring of integer-valued polynomials on D.

Given an ideal $\mathfrak{A}$ of the subring R of $\mathcal{F}(E, D)$, we may, for each $a \in E$, consider the *set of values* of $\mathfrak{A}$, that is,

$$\mathfrak{A}(a) = \{g(a) \mid g \in \mathfrak{A}\}.$$

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Note that, if R contains D (as it is the case, for the ring $\mathrm{Int}(E,D)$ of integer-valued polynomials on E), then $\mathfrak{A}(a)$ is an ideal of D. The problem we address here is: to what extent are the ideals of R characterized by their values? For a given ideal $\mathfrak{A}$, our problem can be rephrased as follows: for $f \in R$, does the condition $f(a) \in \mathfrak{A}(a)$ for each $a \in E$ implies $f \in \mathfrak{A}$? We thus set the following definition.

Definition 1 (Skolem closure)**.** Let E be a set, D be a ring, R be a subring of $\mathcal{F}(E,D)$, and $\mathfrak{A}$ be an an ideal of R. Then

$$\mathfrak{A}^* = \{f \in R \mid f(a) \in \mathfrak{A}(a) \text{ for each } a \in E\}$$

is called *the Skolem closure* of $\mathfrak{A}$. If $\mathfrak{A} = \mathfrak{A}^*$, then $\mathfrak{A}$ is said to be *Skolem closed*.

Note that $\mathfrak{A}^*$ is clearly an ideal of R. Note also that the Skolem closure does indeed behave like a closure [4, Proposition VII.1.6]: $\mathfrak{A}^*$ is the smallest Skolem closed ideal containing $\mathfrak{A}$; an intersection of Skolem closed ideals is Skolem closed; if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A}^* \subseteq \mathfrak{B}^*$.

In particular, if $\mathfrak{A}^* = R$, does it imply that $\mathfrak{A} = R$? (in other words, in the case where R contains D, does $\mathfrak{A}(a) = D$, for each $a \in D$, imply that $\mathfrak{A} = R$?). In the thirties, Thoralf Skolem [23] pointed out that this particular question has a negative answer if R is the ring of polynomials $\mathbb{Z}[X]$ (considered as a ring of functions from $\mathbb{Z}$ to itself): the ideal $\mathfrak{A} = (3, X^2+1)$ is such that $\mathfrak{A}(n) = \mathbb{Z}$ for all n, while $\mathfrak{A} \neq \mathbb{Z}[X]$. However, he proved the answer to be positive for the *finitely generated* ideals of the ring $R = \mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ (which is not Noetherian). We then say that $\mathrm{Int}(\mathbb{Z})$ has the Skolem property. In fact, it was shown by D. Brizolis [1] in the seventies that $\mathrm{Int}(\mathbb{Z})$ has a stronger property: the finitely generated ideals are characterized by their value ideals.

As in the case of $\mathrm{Int}(\mathbb{Z})$, we shall often restrict ourselves to finitely generated ideals. For other reasons (developed below), we shall sometimes also restrict ourselves to *unitary* ideals, that is, ideals containing nonzero constant functions. We then set the following definitions (similar to [4, Definitions VII.1.1 & VII.2.3]):

Definitions 2 (Skolem properties)**.** Let E be a set, D be a ring and R be a subring of $\mathcal{F}(E,D)$.
(i) R is said to satisfy the *Skolem property* (resp., the *almost Skolem property*) if, for each finitely generated ideal (resp., each finitely generated unitary ideal) $\mathfrak{A}$ of R, $\mathfrak{A}^* = R$ implies $\mathfrak{A} = R$.
(ii) R is said to satisfy the *strong Skolem property* (resp., the *almost strong Skolem property*) if each finitely generated ideal (resp., each finitely generated unitary ideal) of R is Skolem closed.
(iii) R is said to satisfy the *super Skolem property* (resp., the *almost super Skolem property*) if each ideal (resp., each unitary ideal) of R is Skolem closed.

The Skolem and strong Skolem properties are classical, the super Skolem property is a new notion. In fact, most papers consider only finitely generated ideals, since in the very classical case of the ring $R = \mathrm{Int}(\mathbb{Z})$ (which satisfies the strong Skolem property), some non-finitely generated ideals are not Skolem closed (as, for instance, the maximal ideal $\mathfrak{M}_x = \{f \in \mathrm{Int}(\mathbb{Z}) \mid f(x) \in p\widehat{\mathbb{Z}_p}\}$, where x belongs to $\widehat{\mathbb{Z}_p}$, the p-adic completion of $\mathbb{Z}$, but not to $\mathbb{Z}$).

Usually the context makes clear in which ring of functions the ring R is contained; let us note, however, that the Skolem properties are relative to it. Indeed, we shall in particular consider the case where R is contained in a ring $\mathcal{C}(E,D)$ of continuous functions (in some topology) and extend these functions to continuous functions from the completion $\widehat{E}$ of E to the completion $\widehat{D}$ of D. Then R may satisfy the super Skolem property as a subring of $\mathcal{F}(\widehat{E}, \widehat{D})$, but only the strong Skolem property as a subring of $\mathcal{F}(E, D)$; indeed, one considers the values at each $x \in \widehat{E}$, in the first case, and only at each $a \in E$, in the second one.

In the first section of this paper, we survey the main known results for the ring $R = \text{Int}(D)$ of integer-valued polynomials on a domain D. This case has been extensively studied by several authors (Brizolis [1], [2], [3], Chabert [8], [9], [10], [11], McQuillan [16], [17], [18]). In particular, we recall the notion of d-ring (equivalent to the Skolem property restricted to non-unitary ideals), and we end this short survey with the study of the almost strong Skolem property in the case of a one-dimensional local Noetherian domain D with finite residue field. We recall that a sufficient condition is that D is *analytically irreducible* (that is, its completion is a domain). We announce also a new result: it is necessary that D is *unibranched*, that is, its integral closure is a local ring.

We then turn to the consideration of the ring $\text{Int}(E, D)$, where E is a subset of the quotient field K of D. In a second section, we study various properties of E, beginning with a generalization of d-rings. In fact, contrary to the case of $\text{Int}(D)$, where various properties are equivalent, we must introduce here several definitions: a d-set is such that each almost integer-valued rational function on E is a polynomial, an s-set is such that each unit-valued polynomial on E is a constant. We show that these notions are distinct. Contrary also to the case of integer-valued polynomials on the ring D, the Skolem closure of a finitely generated ideal $\mathfrak{A}$ of $\text{Int}(E, D)$ is not determined by the ideal of values $\mathfrak{A}(a)$ for a in a cofinite subset of E. Thus we introduce various notions of coherence: we say E is *coherent*, if each almost integer-valued on E is in fact integer-valued, and that E is *strongly coherent*, if for each finitely generated ideal $\mathfrak{A}$ of $\text{Int}(E, D)$, and each polynomial $f \in K[X]$, the condition $f(a) \in \mathfrak{A}(a)$ for almost each $a \in E$, implies $f \in \mathfrak{A}^*$. Again we show that these notions are distinct. Finally, we describe a large class of strongly coherent subsets.

In a third section, we link the Skolem properties to the notion of d-sets and coherence. We first show that, if $\text{Int}(E, D)$ satisfies the almost Skolem property and if E is d-set, then $\text{Int}(E, D)$ satisfies the Skolem property. Turning then to the almost Skolem property, we give an easy generalization of the results on $\text{Int}(D)$. But mainly, we show in this section that $\text{Int}(E, D)$ satisfies the strong Skolem property if and only if it satisfies the almost strong Skolem property and E is a strongly coherent d-set. In the case where $\text{Int}(E, D)$ is a Prüfer domain (for instance if D is a Dedekind domain with finite residue fields), we may even conclude that $\text{Int}(E, D)$ satisfies the strong Skolem property if and only if E is a coherent d-set.

In the last section, we finally turn to the almost strong Skolem property in the local case. As for $\text{Int}(D)$, we consider the case where D is a one-dimensional local Noetherian domain, with maximal ideal $\mathfrak{m}$. We suppose that D is analytically irreducible, but we do not assume that the residue field is finite. Indeed, we show that $\text{Int}(E, D)$ satisfies the almost strong Skolem property if and only if the topological closure $\widehat{E}$ of E (in the $\mathfrak{m}$-adic topology) is compact. We give a couple of examples.

1. The ring $R = \mathrm{Int}(D)$

In this section we recall, without proofs, the known results concerning the ring $R = \mathrm{Int}(D)$ of integer-valued polynomials on a domain D with quotient field K. Most of these results are collected in Chapter VII of [4], to which we send the reader for references.

d-rings. Let us look in particular at the principal ideal (f) generated by a non-constant polynomial $f \in \mathrm{Int}(D)$.
— If $\mathrm{Int}(D)$ satisfies the Skolem property, the Skolem closure of (f) is distinct from $\mathrm{Int}(D)$, and hence, there exists a in D such that $f(a)$ is not a unit in D. We could say that D is an *s-ring* (s for solution): for each non-constant polynomial $f \in \mathrm{Int}(D)$, there exists a maximal ideal $\mathfrak{m}$ of D such that f has a root modulo $\mathfrak{m}$ [4, Proposition VII.2.3].
— If $\mathrm{Int}(D)$ satisfies the strong Skolem property, the ideal (f) is Skolem closed: if a polynomial $g \in \mathrm{Int}(D)$ is such that $g(a) \in f(a)D$ for each $a \in D$ (that is, the rational function g/f takes integer values at each a but possibly the zeros of f), then f divides g in $\mathrm{Int}(D)$ (in particular, g/f is a polynomial). We could say that D is a *d-ring* (d for divisibility): each rational function which is integer-valued for almost all $a \in D$ is in fact an integer-valued polynomial.

Obviously, a d-ring is an s-ring, but as it turns out these properties are equivalent [4, Proposition VII.2.3]. For the s-ring property, we may consider only polynomials with coefficients in D [4, Exercise VII.7]. Recall also that an almost integer-valued rational function (that is, a function which takes integer-values for each $a \in D$, but finitely many), is in fact integer-valued [4, Lemma VII.1.8]. Thus, we list below several equivalent statements for the definition of a d-ring.

Proposition-Definition 1.1. *We say that a domain D is a* d-ring *if it satisfies the following equivalent conditions*:

(i) *each integer-valued rational function on D is a polynomial,*
(ii) *each almost integer-valued rational function on D is an integer-valued polynomial,*
(iii) *for each non-constant polynomial $f \in D[X]$ (resp., each non-constant polynomial $f \in \mathrm{Int}(D)$), there exists $a \in D$ such that $f(a)$ is not a unit of D,*
(iv) *for each non-constant polynomial $f \in D[X]$ (resp., each non-constant polynomial $f \in \mathrm{Int}(D)$), the intersection of the maximal ideals $\mathfrak{m}$ of D for which f has a root modulo $\mathfrak{m}$ is* (0).

There are many examples of d-rings (for instance, see [14]): in particular, every domain which is a finitely generated $\mathbb{Z}$-algebra is a d-ring. However it is easy to note that this property is not stable under localization; in fact, a semi-local domain is never a d-ring (consider the polynomial $1 + mX$, where m is in the Jacobson radical of D).

As seen at the beginning of this section, D is a d-ring if and only if each principal ideal (f) of $\mathrm{Int}(D)$ is Skolem closed. In fact, we may then restrict our attention to unitary ideals (that is, to the almost Skolem properties), indeed we have the following [4, Proposition VII.2.14]:

Proposition 1.2. *Let D be a domain. Then* Int(D) *satisfies the Skolem property* (*resp., the strong Skolem property*) *if and only if*

a) *D is a d-ring,*
b) Int(D) *satisfies the almost Skolem property* (*resp., the almost strong Skolem property*).

Divisorial ideals and Prüfer domains. Recall that a *divisorial* ideal is an intersection of principal fractional ideals. For a d-ring D, each principal ideal, thus, each divisorial ideal of Int(D) is Skolem closed (since an intersection of Skolem closed ideals is Skolem closed). Restricting ourselves to unitary ideals, we have the following, without any hypothesis on D [4, Lemma VII.2.15]:

Lemma 1.3. *Let D be a domain. Each unitary divisorial ideal of* Int(D) *is Skolem closed.*

If Int(D) is a Prüfer domain, each finitely generated ideal is divisorial. Moreover, if D is a Noetherian domain, we know that Int(D) is a Prüfer domain if and only if D is a Dedekind domain with finite residue fields [4, Theorem VI.1.7]. Hence, we may summarize ourselves with the following.

Proposition 1.4. *Let D be a domain such that* Int(D) *is a Prüfer domain* (*for instance, a Dedekind domain with finite residue fields*), *then* Int(D) *satisfies the almost strong Skolem property. Moreover, the following assertions are equivalent*:

(i) *D is a d-ring,*
(ii) Int(D) *satisfies the Skolem property,*
(iii) Int(D) *satisfies the strong Skolem property.*

In particular, if D is the ring of integers of a number field, then Int(D) satisfies the strong Skolem property.

Using divisorial ideals, we arrived easily at a conclusion for the class of Dedekind domains. In fact, we can characterize the Noetherian domains D such that Int(D) satisfies the Skolem property, and give necessary or sufficient conditions for the strong Skolem property. We will see that D need not be integrally closed.

Almost Skolem property. We have the following characterization of the almost Skolem property in the Noetherian case [4, Proposition VII.4.5].

Proposition 1.5. *Assume D is Noetherian* (*but not a field*). *Then* Int(D) *satisfies the almost Skolem property if and only if*

a) *for each maximal ideal $\mathfrak{m}$ of D, either $D/\mathfrak{m}$ is algebraically closed, or $\mathfrak{m}$ is an height-one prime and $D/\mathfrak{m}$ is finite,*
b) *each nonzero prime ideal of D is an intersection of maximal ideals.*

Let us emphasize in particular that every one-dimensional Noetherian domain, with finite residue fields, satisfies the almost Skolem property. We would derive immediately a characterization of the Skolem property adding the condition that D is a d-ring.

Remarks 1.6. (1) If D is a d-ring then, in particular, the ideal (0) is an intersection of maximal ideals [Proposition-Definition 1.1]. Thus if Int(D) satisfies the Skolem property we derive that D is an Hilbert ring (that is, each prime ideal is an intersection of maximal ideals).

(2) Brizolis [1] pointed out a property which seems a priori stronger than the Skolem property, that he called the *Nullstellensatz property*: the Skolem closure of each finitely generated ideal $\mathfrak{A}$ of $\mathrm{Int}(D)$ is contained in the radical $\sqrt{\mathfrak{A}}$ of $\mathfrak{A}$. In fact, the Nullstellensatz and Skolem properties are equivalent in the Noetherian case [4, Proposition VII.4.5].

Almost strong Skolem property. For the almost strong Skolem property, we first have a necessary condition [4, Proposition VII.3.3].

Lemma 1.7. *Assume D is Noetherian (but not a field). If $\mathrm{Int}(D)$ satisfies the almost strong Skolem property, then D is one-dimensional with finite residue fields.*

Moreover, contrary to the Skolem properties, the almost strong Skolem property is local in the Noetherian case [4, Exercise VII.18]:

Lemma 1.8. *Assume D is Noetherian. Then $\mathrm{Int}(D)$ satisfies the almost strong Skolem property if and only if, for each maximal ideal $\mathfrak{m}$ of D, $\mathrm{Int}(D_{\mathfrak{m}})$ satisfies the almost strong Skolem property.*

Thus, we restrict ourselves to a one-dimensional Noetherian local domain D, with maximal ideal $\mathfrak{m}$, and we assume its residue field to be finite. Recall that D is said to be *analytically irreducible* if its completion $\widehat{D}$ in the $\mathfrak{m}$-adic topology is an integral domain, and that D is said to be *unibranched* if its integral closure D' is local (that is, a rank-one discrete valuation domain). It is known that an analytically irreducible domain is unibranched, while the converse does not hold in general. These conditions are linked to the almost strong Skolem property, and we may summarize what we know as follows:

Theorem 1.9. *Let D be a one-dimensional local Noetherian domain, with finite residue field.*

- (i) *If D is analytically irreducible, then $\mathrm{Int}(D)$ satisfies the almost strong Skolem property.*
- (ii) *If $\mathrm{Int}(D)$ satisfies the almost strong Skolem property, then D is unibranched.*

Let us give some comments on both assertions.
(i) We knew the first condition (D is analytically irreducible) to be sufficient [4, Theorem VII.3.8] (the result is given there in more generality). The proof is based on the following facts, using the compactness of $\widehat{D}$:
— A. The ring $\mathcal{C}(\widehat{D},\widehat{D})$ satisfies the almost super Skolem property: the unitary ideals (not only the finitely generated unitary ideals) are characterized by their value ideals on $\widehat{D}$.
— B. Analogously to the classical Stone-Weierstrass theorem, $\mathrm{Int}(D)$ is dense in $\mathcal{C}(\widehat{D},\widehat{D})$ for the uniform convergence topology [4, Theorem III.5.3]. It follows that, as a subring of $\mathcal{F}(\widehat{D},\widehat{D})$, $\mathrm{Int}(D)$ satisfies also the almost super Skolem property (in fact, for each unitary ideal $\mathfrak{A}$ of $\mathrm{Int}(D)$, if $f(x)\in\mathfrak{A}(x)\widehat{D}$ for each $x\in\widehat{D}$, then $f\in\mathfrak{A}$) [4, Theorem VII.3.7].
— C. For each finitely generated unitary ideal $\mathfrak{A}$ of $\mathrm{Int}(D)$, the ideals of values are locally constant on D: there exists a nonzero ideal $\mathfrak{b}$ of D such that $(a-b)\in\mathfrak{b}$ implies $\mathfrak{A}(a)=\mathfrak{A}(b)$ [4, Lemma VII.1.9]. It follows that $f(a)\in\mathfrak{A}(a)$ for each $a\in D$ implies that $f(x)\in\mathfrak{A}(x)\widehat{D}$ for each $x\in\widehat{D}$.
(ii) We proved recently that the second condition (D is unibranched) is necessary [a paper is under preparation]: if D is not unibranched, there exists a nonzero ideal $\mathfrak{a}$

in D, such that the ideal $\mathrm{Int}(D, \mathfrak{a})$ (of integer-valued polynomials with values in $\mathfrak{a}$) is not finitely generated. However, it is the Skolem closure of the finitely generated ideal $\mathfrak{a}\mathrm{Int}(D)$.

Question 1.10. If D is a one-dimensional local Noetherian domain, with finite residue field, we know that D is analytically irreducible if and only if $\mathrm{Int}(D)$ satisfies the Stone-Weierstrass property (that is, is dense in $\mathcal{C}(\widehat{D}, \widehat{D})$). On the other hand, D is unibranched if and only if $\mathrm{Int}(D)$ satisfies the following *interpolation property*: for each finite set $(a_1, \ldots, a_n)$ of distinct elements of D, and each corresponding set of "values" $(c_1, \ldots, c_n)$ in D, there exists $f \in \mathrm{Int}(D)$ such that $f(a_i) = c_i$ for $1 \leq i \leq n$ [6, Theorem 3.1]. Is the almost strong Skolem property equivalent to one of these properties? In general, what relation is there between the almost strong Skolem property and the interpolation property?

Various generalizations. More generally, we may consider fractional ideals of $\mathrm{Int}(D)$ (as in [5]), and see to what extent they are characterized by their values. If $\mathfrak{A}$ is a fractional ideal, there is a polynomial f such that $f\mathfrak{A}$ is an integral ideal. It follows that, if $\mathrm{Int}(D)$ satisfies the super Skolem property (resp., the strong Skolem property), the fractional ideals (resp. the finitely generated fractional ideals) are characterized by their values: if $\mathfrak{A}$ is a fractional ideal (resp., a finitely generated fractional ideal), and φ a rational function such that $\varphi(a) \in \mathfrak{A}(a)$ for almost all $a \in \mathfrak{A}$, then $\varphi \in \mathfrak{A}$ (in particular, letting $\mathfrak{A} = \mathrm{Int}(D)$, we recover the fact that D must be a d-ring).

We may also consider the ring of integer-valued polynomials in n indeterminates:

$$R = \mathrm{Int}(D^n) = \{f \in K[X_1, \ldots, X_n] \mid f(D^n) \subseteq D\}.$$

For $n > 1$, the ring $\mathrm{Int}(D^n)$ is never a Prüfer domain (even if D is a Dedekind domain with finite residue fields); nevertheless, we still have the following [4, Proposition XI.3.8]:

Proposition 1.11. *Let D be a* (*local, one-dimensional*) *analytically irreducible Noetherian domain with finite residue field. Then $\mathrm{Int}(D^n)$ satisfies the almost strong Skolem property.*

We could finally consider the ring of integer-valued rational functions:

$$\mathrm{Int}^R(D) = \{\varphi \in K(X) \mid \varphi(D) \subseteq D\}.$$

Even in the local case, although D is not a d-ring, we may have the strong Skolem property for the ring of rational functions. Indeed, we then have [4, Proposition X.3.8]:

Proposition 1.12. *Let D be a* (*local, one-dimensional*) *analytically irreducible Noetherian domain with finite residue field. Then $\mathrm{Int}^R(D)$ satisfies the strong Skolem property.*

2. d-sets, coherent and homogeneous subsets

In this section, we let D be a domain with quotient field K, and we consider a subset E of K (which is not necessarily a subset of D). We consider the ring $\mathrm{Int}(E, D)$ of *integer-valued polynomials on E*:

$$\mathrm{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}.$$

We study various properties of the subset E, related to the Skolem properties of $\mathrm{Int}(E, D)$, which allow, in the next sections, to restrict ourselves to the almost Skolem properties.

s-sets and d-sets. As in the case of $\mathrm{Int}(D)$, let us first examine the Skolem properties with respect to a principal ideal (f), where f is a non-constant polynomial in $\mathrm{Int}(E, D)$.
— If $\mathrm{Int}(E, D)$ satisfies the Skolem property, the Skolem closure of (f) is distinct from $\mathrm{Int}(E, D)$, and hence, there exists a in D such that $f(a)$ is not a unit in D.
— If $\mathrm{Int}(E, D)$ satisfies the strong Skolem property, the ideal (f) is Skolem closed: if a polynomial $g \in \mathrm{Int}(E, D)$ is such that $g(a) \in f(a)D$ for each $a \in D$ (that is, g/f is an almost integer-valued rational function on E), then f divides g in $\mathrm{Int}(E, D)$ (in particular, g/f is a polynomial).

We thus set the following definitions (the first one is in [12]).

Definitions 2.1. (i) We say that a subset E of K is a *d-set* (*with respect to D*) if each almost integer-valued rational function on E is a polynomial.
(ii) We say that E is an *s-set* (*with respect to D*) if each unit-valued polynomial on E is a constant.

If the context is clear, we may drop the reference to D. If E is a subset of D, we may also say that E is a d-subset (resp., an s-subset) of D.

If f is a unit-valued polynomial on E, then $(1/f)$ is an integer-valued rational function on E; also, if f is a unit-valued polynomial (resp., if φ is an integer-valued rational function) on some subset F of K, then a fortiori f is unit-valued (resp., φ is integer-valued) on a subset E of F. We thus have immediately the following properties.

Proposition 2.2. *Let E be a subset of K.*

(i) *If E is a d-set, then E is an s-set.*
(ii) *If E is a d-set (resp., an s-set), and if $E \subseteq F$, then F is also a d-set (resp., an s-set).*

In particular, if there is an s-subset in D, then D is an s-subset of itself and, in fact, a d-ring [Proposition-Definition 1.1]. We give below an example showing that, contrary to the case of $\mathrm{Int}(D)$, the notions of d-set and of s-set are not equivalent [Example 2.4 (4)]. But first, we make some comments.

Remarks 2.3. (1) To say that E is not a d-set means that we can find an almost integer-valued rational function φ, which is not a polynomial. In fact, we may ask φ to be of the form $\varphi = d/g$, where $d \in D$ is a nonzero constant and $g \in D[X]$ is a non-constant polynomial with coefficients in D. Indeed, write $\varphi = f/g$, where f and g are polynomials of $D[X]$ which are coprime in $K[X]$. We then have $uf + vg = d$, where $u, v \in D[X]$, and $d \in D, d \neq 0$. Clearly, d/g is also almost integer-valued.
(2) To say that E is not an s-set means that we can find a non-constant polynomial f which is unit-valued on E. Necessarily $f \in \mathrm{Int}(E, D)$, but that does not mean that we can find such a polynomial with coefficients in D. We could thus say that E is a *weak s-set*, if each polynomial $f \in D[X]$, which is unit-valued on E, is constant. Contrary to the case of $\mathrm{Int}(D)$, a weak s-set need not be an s-set [Example 2.4 (5)].

Examples 2.4. (1) *Finite subsets.* A finite set $E = \{a_1, \ldots, a_r\}$ is never an s-set. Indeed, the polynomial $f = 1 + \prod_{i=1}^{r}(X - a_i)$ is unit-valued on E.

(2) *Subsets of* $\mathbb{Z}$. A subset of $\mathbb{Z}$ is a d-subset (resp., an s-subset) if and only if it is infinite [22, vol. II.8/II.93]. Indeed, the condition is necessary from the previous example. It is sufficient: if φ is an integer-valued rational function on an infinite subset of $\mathbb{Z}$, write $\varphi = f + p/q$, where f, p, q are polynomials with coefficients in $\mathbb{Q}$, and $\deg(p) < \deg(q)$. Let $d \in \mathbb{Z}, d \neq 0$, be such that $df \in \mathbb{Z}[X]$. Then, for each $x \in \mathbb{Z}, dp(x)/q(x) \in \mathbb{Z}$, and moreover converges to 0 as x goes to infinity. Necessarily, $p = 0$.
(3) *Finite group of units.* If the group of units of D is finite, every infinite set E is an s-set: if a polynomial is unit-valued on E, it takes finitely many values on an infinite set, thus it is a constant.
(4) *An s-set which is not a d-set.* Let $D = \mathbb{Z} + t\mathbb{Q}[t]$ be the ring formed by the polynomials with coefficients in $\mathbb{Q}$ and constant term in $\mathbb{Z}$, and $E = \mathbb{Z}$. Clearly, E is not a d-subset of D, since the rational function $\frac{t}{1+X^2}$ is integer-valued on $E = \mathbb{Z}$. On the other hand, the units of D are 1 and -1, and it follows from the previous example that E is an s-subset of D.
(5) *A subset E which is not an s-set, but such that each polynomial, with coefficients in D, which is unit-valued on E, is constant.* Let $D = \mathbb{Q}[t]$ be the ring of polynomials with coefficients in $\mathbb{Q}$, and $E = \{at \mid a \in \mathbb{Q}\}$ be the subset of D formed by the monomials of degree one. Clearly, E is not an s-subset, since the polynomial $\frac{X^2+t^2}{t^2}$ is unit-valued on E. Now, let $f \in D[X]$. We can write

$$f = h_0(t) + h_1(t)X + \ldots + h_n(t)X^n,$$

where each $h_i(t)$ is a polynomial with coefficients in $\mathbb{Q}$. Suppose that f is unit-valued: for each $a \in \mathbb{Q}$, $f(at)$ is a unit, that is, an element of $\mathbb{Q}$. Write

$$f(Xt) - f(0) = t\big[h_1(t)X + \ldots + h_n(t)t^{n-1}X^n\big].$$

Thus, $f(at) - f(0)$ is an element of $\mathbb{Q}$ which is divisible by t, and hence, is null for each $a \in \mathbb{Q}$. Since $\mathbb{Q}$ is infinite, the polynomial $f(Xt) - f(0)$ is identically null, whence so is each $h_i(t)$. Finally, $f = h_0(t)$ is a constant (that is, an element of D).

Coherent and strong coherent sets. Contrary to the case of $\mathrm{Int}(D)$, the Skolem closure of a finitely generated ideal $\mathfrak{A}$ of $\mathrm{Int}(E, D)$ is not necessarily determined by the ideal of values $\mathfrak{A}(a)$ for a in a cofinite subset of E; a polynomial f may even be such that $f(a) \in D$, for almost all $a \in E$, but not be integer-valued on E. This is obviously the case if E is finite, but here is a less trivial example: let $\mathbb{P}$ be the set of prime numbers; for each prime number p, the polynomial $\frac{(X-1)\cdots(X-p+1)}{p}$ is clearly integer-valued on each prime number but p. We then set the following definitions.

Definitions 2.5. (i) We say that a subset E of K is *coherent* (*with respect to* D) if each polynomial $f \in K[X]$ which is almost integer-valued on E is in fact integer-valued on E.
(ii) We say that E is *strongly coherent* (*with respect to* D) if, for each finitely generated ideal $\mathfrak{A}$ of $\mathrm{Int}(E, D)$, each polynomial $f \in K[X]$ such that $f(a) \in \mathfrak{A}(a)$ for almost each $a \in E$, belongs to the Skolem closure $\mathfrak{A}^*$ of $\mathfrak{A}$.

As for d-and s-sets, we often drop the reference to D. Note that a ring D is always a strongly coherent subset of itself [4, Lemma VII.1.8]. Note also that we restricted ourselves to finitely generated ideals of $\mathrm{Int}(E, D)$. Indeed, consider the maximal ideal $\mathfrak{M}_{p,n} = \{f \in \mathrm{Int}(\mathbb{Z}) \mid f(n) \in p\mathbb{Z}\}$ of $\mathrm{Int}(\mathbb{Z})$, where p is a prime number and n an integer: the constant polynomial $f = 1$ is such that $f(a) \in \mathfrak{M}_{p,n}(a)$, for each $a \neq n$, while $f(n) \notin \mathfrak{M}_{p,n}(n)$.

A strongly coherent set is coherent (consider the ideal $\mathfrak{A} = \mathrm{Int}(E, D)$). The following example shows that the converse does not hold.

Example 2.6. *A coherent set which is not strongly coherent.* Let $D = \mathbb{C}[t]$ be the ring of polynomials with coefficients in the complex field $\mathbb{C}$, and let $E = \mathbb{C}$. By a Vandermonde argument, it is easy to see that, for each infinite subset F of E, we have $\mathrm{Int}(F, D) = D[X]$ [4, Proposition I.3.1]. We obtain the following containments, thus, in fact, equalities

$$D[X] \subseteq \mathrm{Int}(D) \subseteq \mathrm{Int}(E, D) \subseteq \mathrm{Int}(F, D) \subseteq D[X].$$

In particular, E is coherent: each polynomial which is integer-valued on a cofinite subset F of E is, in fact, integer-valued on E. On the other hand, the polynomial X is clearly unit-valued on the complement of 0 on E, but not on E. Letting $\mathfrak{A} = (X)$ be the ideal generated by X, and $f = 1$, we then have $f(a) \in \mathfrak{A}(a)$ for all $a \in E$, but 0. And hence, E is not strongly coherent.

Remarks 2.7. (1) Extending [4, Definitions IV.1.2], we may say that a subset F of E is *polynomially dense* in E if $\mathrm{Int}(F, D) = \mathrm{Int}(E, D)$. To say that E is coherent thus means that each cofinite subset of E is polynomially dense in E. In fact, we could rather say that E is *polynomially coherent* in this case. We could then similarly say that F is *rationally dense* in E, if each rational function which is integer-valued on F is in fact integer-valued on E, and that E is *rationally coherent* if each cofinite subset of E is rationally dense in E. Example 2.6 shows that these two notions are distinct: the rational function $\frac{1}{X}$ is unit-valued on the complement of 0 on E.
(2) A strongly coherent set is even rationally coherent. This follows immediately from the fact that we can extend the property of the definition to rational functions. Suppose indeed that $\varphi(a) \in \mathfrak{A}(a)$ for almost each $a \in E$. Write $\varphi = f/g$, where f and g are integer-valued and coprime in $K[X]$, and set $\mathfrak{B} = g\mathfrak{A}$. If $\varphi(a) \in \mathfrak{A}(a)$, then $f(a) \in \mathfrak{B}(a)$ for almost each a. If E is strongly coherent, it follows that $f(a) \in \mathfrak{B}(a)$ for all a. In particular, if $g(a) = 0$, then $f(a) = 0$, but since f and g are coprime, this never happens: $g(a)$ never vanishes. In conclusion $\varphi(a) = f(a)/g(a)$ belongs to $\mathfrak{A}(a)$ for all a. We do not know if the notions of strong and rational coherence are distinct (for instance, we shall see that they are equivalent if $\mathrm{Int}(E, D)$ is a Prüfer domain [Proposition 3.10]).
(3) As in [12, Definition B], we could say that an element $a \in E$ is (*polynomially*) *isolated* in E if its complement is not polynomially dense in E (for instance, we saw above that each point of $\mathbb{P}$ is isolated in $\mathbb{P}$). Similarly we could say that $a \in E$ is *rationally isolated* in E if its complement is not rationally dense in E. A polynomially isolated point is rationally isolated; the converse does not hold (in Example 2.6 above, the point 0 is rationally, but not polynomially isolated). Of course a coherent (resp., a rationally coherent) set cannot have any isolated (resp., rationally isolated) point. We do not know if the converse holds, thus we end this paragraph with a question.

Question 2.8. Suppose that E is a subset of K without isolated points (resp., without rationally isolated points). Does this imply that E is coherent (resp., rationally coherent)?

We have a partial answer: if E is a subset of a completely integrally closed domain D, the following assertions are equivalent [12, Corollary 2.2 & Theorem 3.1], [4, Exercise VI.10].

— E is coherent,
— E has no isolated point,
— $\mathrm{Int}(E, D)$ is completely integrally closed.

Coherent d-sets. The notion of d-set or s-set on one-hand, and of coherence on the other, are clearly distinct. For instance, a ring D is always a strongly coherent subset of itself, but not necessarily an s-set (that is, a d-ring). On the other hand, the set $\mathbb{P}$ of prime numbers is infinite, and hence, a d-subset of $\mathbb{Z}$ [Example 2.4 (2)], but it is not coherent. We shall need the (immediate) following characterization of the cases where both properties hold.

Proposition 2.9. *A subset E of K is a a coherent d-set if and only if each almost integer-valued rational function on E is an integer-valued polynomial.*

Remark 2.10. We could say that a subset E of K is an *almost d-set* if each integer-valued rational function on E is a polynomial (and of course, such a polynomial is then integer-valued). If E is a d-set, each almost-integer valued rational function is a polynomial, but note that here, such a polynomial need not be integer-valued (unless E is coherent). Clearly, E is a coherent d-set if and only if it is a rationally coherent almost d-set. The following example (in fact, Example 2.6) shows that an almost d-set need not be a d-set (even if it is coherent).

Example 2.11. As in Example 2.6, let $D = \mathbb{C}[t]$ be the ring of polynomials with complex coefficients, and let $E = \mathbb{C}$. Then E is not a d-set: the rational function $\frac{1}{X}$ is almost integer-valued, yet not a polynomial. We have seen that E is coherent. Finally, we show that E is an almost d-set: each integer-valued rational function is a polynomial. From Remark 2.3 (1), it suffices to show that, if $d/g(X)$ is integer-valued, where d is a nonzero element of D, and $g(X)$ is a polynomial with coefficients in D, then $g(X)$ is a constant, that is, an element of D. Write $\frac{d}{g(X)} = \frac{d(t)}{g(t,X)}$, where $d(t)$ is a polynomial with coefficients in $\mathbb{C}$, and

$$g(t, X) = g_0(t) + g_1(t)X + \ldots + g_n(t)X^n,$$

each $g_i(t)$ being also a polynomial with coefficients in $\mathbb{C}$. To say that $d/g(X)$ is integer-valued means that, for each $a \in E = \mathbb{C}$, $d/g(a)$ belongs to $D = \mathbb{C}[t]$, that is,

$$d(t) = h_a(t)g(t, a) \text{ where } h_a(t) \in \mathbb{C}[t].$$

If, for some $\alpha \in \mathbb{C}$, the polynomial $g(\alpha, X)$ is not constant, it has a root in $\mathbb{C}$: there is $a \in \mathbb{C}$ such that $g(\alpha, a) = 0$, and hence, such that $d(\alpha) = 0$. Since $d(t)$ has only finitely many roots in $\mathbb{C}$, this implies that $g(\alpha, X)$ is a constant, for almost each $\alpha \in \mathbb{C}$. Since $g(\alpha, X) = g_0(\alpha) + g_1(\alpha)X + \ldots + g_n(\alpha)X^n$, it follows that $g_i(t) = 0$ for $i \geq 1$. Therefore $g(t, X) = g_0(t)$, that is, $g(t, X)$ is an element of $D = \mathbb{C}[t]$.

Homogeneous and weakly-homogeneous sets. We describe here a class of strongly coherent sets. It generalizes the *homogeneous* subsets (in a Dedekind domain) of D. McQuillan, which are union of cosets modulo some nonzero ideal [20].

Definitions 2.12. (i) We say that a subset E of K is *homogeneous* (*with respect to D*), if there exists a nonzero ideal $\mathfrak{a}$ of D such that $(a + \mathfrak{a}) \subseteq E$ for each $a \in E$.
(ii) We say that E is *weakly-homogeneous* (*with respect to D*) if, for each nonzero ideal $\mathfrak{a}$ of D, and each $a \in E$, $(a + \mathfrak{a})$ contains at least one element of E distinct from a.

As usual, we shall often drop the reference to D. If E is homogeneous, and if α is a nonzero element of the ideal $\mathfrak{a}$, then clearly $(a+\alpha D) \subseteq E$. Thus we could define homogeneous sets using only principal ideals. The same holds for weakly-homogeneous sets.

Let us now introduce some topological ideas. We can consider the topology where the nonzero ideals form a basis of neighborhoods of 0, let us call it the *ideal topology*. It follows from [4, Lemma I.3.19] that each polynomial is uniformly continuous in this topology. (We could derive that a topologically dense subset F of E is polynomially dense; however the converse does not hold: a subset of $\mathbb{Z}$ is polynomially dense in $\mathbb{Z}$ if and only if it is dense in every p-adic topology, which does not imply that it is dense in the ideal topology [4, Remark IV.2.8].) To say that E is weakly homogeneous means that each element of E is an accumulation point of E in the ideal topology. In particular, for each nonzero ideal $\mathfrak{a}$ of D, and each $a \in E$, the intersection $(a+\mathfrak{a}) \cap E$ is infinite. We leave to the reader that a homogeneous set is weakly homogeneous. It is easy to see that the union of the intervals $[n!, n!+n]$ is weakly homogeneous in $\mathbb{Z}$ (since it is dense in $\mathbb{Z}$, in the ideal topology), but not homogeneous (since it does not contain any arithmetic progression).

We show now that a weakly homogeneous set is strongly coherent; this is very similar to the proof that D itself (which obviously is homogeneous) is strongly coherent [4, Lemma VII.1.8].

Proposition 2.13. *A weakly homogeneous set is strongly coherent.*

Proof. Let $\mathfrak{A}$ be a finitely generated ideal of $\mathrm{Int}(E,D)$, and f be a polynomial such that $f(a) \in \mathfrak{A}(a)$ for almost each $a \in E$. Let b be an element of E, we wish to show that $f(b) \in \mathfrak{A}(b)$. Denote by $g_1, \ldots, g_r$ a set of generators of $\mathfrak{A}$. The polynomials f and g_j have a common denominator d. Whatever $a \in E$, $f(a) \in \mathfrak{A}(a)$ if and only if $df(a) \in d\mathfrak{A}(a)$. With no loss of generality, we may thus assume the polynomials f and g_j to have their coefficients in D. Since E is weakly homogeneous, we may choose, for each nonzero ideal $\mathfrak{a}$, an element $a \in E$ of the form $a = b + \alpha$, where $\alpha \in \mathfrak{a}$, such that $f(a) \in \mathfrak{A}(a)$. We then have

- $f(b) = f(a) + \beta$, where $\beta \in \mathfrak{a}$, and
- $g_j(b) = g_j(a) + \beta_j$, where $\beta_j \in \mathfrak{a}$, for $1 \leq j \leq r$.

We then consider two cases:
— $\mathfrak{A}(b) = (0)$. Then $g_j(b) = 0$ for $1 \leq j \leq r$. For each choice of the ideal $\mathfrak{a}$, we have $g_j(a) \in \mathfrak{a}$, for $1 \leq j \leq r$. Hence $\mathfrak{A}(a) \subseteq \mathfrak{a}$. In particular $f(a) \in \mathfrak{a}$, and finally $f(b) \in \mathfrak{a}$. It follows that $f(b) = 0$ (since $f(b)$ belongs to every nonzero ideal of D).
— $\mathfrak{A}(b) \neq (0)$. Choose $\mathfrak{a} = \mathfrak{A}(b)$. We then have $g_j(a) \in \mathfrak{A}(b)$ for $1 \leq j \leq r$. Hence $\mathfrak{A}(a) \subseteq \mathfrak{A}(b)$. In particular $f(a) \in \mathfrak{A}(b)$, and finally $f(b) \in \mathfrak{A}(b)$. □

Isomorphic subsets. We consider two elements $\alpha, \beta, \in K$, such that $\beta \neq 0$, and the K-isomorphism $\Psi_{\alpha,\beta} : K(X) \mapsto K(X)$ defined by $\Psi_{\alpha,\beta}(X) = \frac{X-\alpha}{\beta}$. For each subset E of K, we then denote by $E_{\alpha,\beta}$ the subset $E_{\alpha,\beta} = \alpha + \beta E$. With these notations, we obviously have the following isomorphisms.

Lemma 2.14. *The K-isomorphism $\Psi_{\alpha,\beta}$ induces an isomorphism from* $\mathrm{Int}(E,D)$ *onto* $\mathrm{Int}(E_{\alpha,\beta}, D)$.

We can consider that the subsets E and $E_{\alpha,\beta} = (\alpha + \beta E)$ are, in a sense, isomorphic, hence they share many properties:

Proposition 2.15. (i) *E is an s-set* (*resp., a d-set*) *if and only if $E_{\alpha,\beta}$ is an s-set* (*resp., a d-set*).

(ii) *E is coherent* (*resp., strongly coherent*) *if and only if $E_{\alpha,\beta}$ is coherent* (*resp., strongly coherent*).

(iii) *E is homogeneous* (*resp., weakly homogeneous*) *if and only if $E_{\alpha,\beta}$ is homogeneous* (*resp., weakly homogeneous*).

(iv) $\mathrm{Int}(E, D)$ *has some Skolem property if and only if $E_{\alpha,\beta}$ has the same Skolem property.*

Proof. Only the assertions dealing with ideals (that is, the strong coherence and the Skolem properties) deserve an explanation. For each ideal $\mathfrak{A}$ of $\mathrm{Int}(E, D)$, and each $a \in E$, $\mathfrak{A}(a) = \Psi_{\alpha,\beta}(\mathfrak{A})(\alpha + \beta a)$. It follows that $g(a) \in \mathfrak{A}(a)$ if and only if $\Psi_{\alpha,\beta}(g)(\alpha + \beta a) \in \Psi_{\alpha,\beta}(\mathfrak{A})(\alpha + \beta a)$; in other words, $\left(\Psi_{\alpha,\beta}(\mathfrak{A})\right)^* = \Psi_{\alpha,\beta}(\mathfrak{A}^*)$. □

Remarks 2.16. (1) The morphism $\Psi_{\alpha,\beta}$ does not necessarily take polynomials with coefficients in D into polynomials with coefficients in D. Thus, for instance, the subset $E = \{at \mid a \in \mathbb{Q}\}$ of $D = \mathbb{Q}[t]$ is such that each unit-valued polynomial with coefficients in D is constant [Example 2.4 (5)], while the polynomial $X^2 + 1$ is unit-valued on $\mathbb{Q} = (1/t)E$.
(2) When studying the ring $\mathrm{Int}(E, D)$, one often assumes that E is a *fractional subset* of D, that is, a subset of K such that $dE \subseteq D$ for a nonzero element d of D (for instance, if D is integrally closed, this condition is necessary for $\mathrm{Int}(E, D)$ to contain nonzero constants [4, Corollary I.1.10]). It follows from the previous proposition, that we may often restrict ourselves to subsets of D (a fractional subset is isomorphic to a subset of D having similar properties).

We finally recover and generalize [12, Lemma 2.6].

Corollary 2.17. *If E is a homogeneous subset of a d-ring, then E is a d-set.*

Proof. By hypothesis, E contains a subset of the form $\alpha + \beta D$, which is isomorphic to D, and hence, which is a d-set. Then E itself is a d-set [Proposition 2.2]. □

3. Coherence and Skolem properties

In this section, we consider again a subset E of K, and link the Skolem properties to the notion of d-sets and coherence. We begin with the Skolem property, then proceed to the strong Skolem property. But first, we examine the case where E is finite.

Finite sets. A finite set E is never an s-set [Example 2.4 (1)], thus $\mathrm{Int}(E, D)$ does not satisfy the Skolem property. Nevertheless, let us recall the following result of D. L. McQuillan [19].

Proposition 3.1. *Let E be a non-empty finite set, then* $\mathrm{Int}(E, D)$ *satisfies the almost super Skolem property.*

Proof. Let $E = \{a_1, \ldots, a_r\}$ and set $\psi = \prod_{i=1}^r (X - a_i)$. Consider a unitary ideal $\mathfrak{A}$ and a polynomial f in its Skolem closure $\mathfrak{A}^*$. Then $f(a_i) = g_i(a_i)$, where $g_i \in \mathfrak{A}$ for $1 \leq i \leq r$. Using Lagrange interpolation, we may write $f = \sum_{i=1}^r \varphi_i g_i + g$, where $\varphi_i = \prod_{j \neq i} \frac{X - a_j}{a_i - a_j}$, and thus, g vanishes on E. Since $\mathfrak{A}$ is unitary, it contains a nonzero constant a. Writing $g = a(g/a)$, it is clear that (g/a) vanishes also on E. Hence we have $(g/a) \in \mathrm{Int}(E, D)$, thus $g \in \mathfrak{A}$, and finally, $f \in \mathfrak{A}$. □

Skolem property. In the previous section, considering a principal ideal (f), where f is a non-constant polynomial in $\mathrm{Int}(E,D)$, we saw that, if $\mathrm{Int}(E,D)$ satisfies the Skolem property, then E must be an s-subset. Clearly, $\mathrm{Int}(E,D)$ must also satisfy the almost Skolem property. Contrary to the case of $\mathrm{Int}(D)$ [Proposition 1.2], we have only a partial converse.

Proposition 3.2. *Let E be a subset of K.*

(i) *If* $\mathrm{Int}(E,D)$ *satisfies the Skolem property, then E is an s-set.*
(ii) *If E is a d-set, and if* $\mathrm{Int}(E,D)$ *satisfies the almost Skolem property, then* $\mathrm{Int}(E,D)$ *satisfies the Skolem property.*

The first statement is immediate, and (ii) follows from the next lemma (which implies in particular that the Skolem closure of a non-unitary finitely generated ideal is a proper ideal of $\mathrm{Int}(E,D)$).

Lemma 3.3. *Let E be a d-set of D. If $\mathfrak{A}$ is a finitely generated ideal of* $\mathrm{Int}(E,D)$, *such that $\mathfrak{A} \subseteq \big(gK[X] \cap \mathrm{Int}(E,D)\big)$, for some non-constant polynomial g, then the Skolem closure $\mathfrak{A}^*$ of $\mathfrak{A}$ is also such that $\mathfrak{A}^* \subseteq \big(gK[X] \cap \mathrm{Int}(E,D)\big)$.*

Proof. Since $\mathfrak{A}$ is finitely generated, there is a nonzero element $d \in D$ such that $d\mathfrak{A} \subseteq g\mathrm{Int}(E,D)$. Let $f \in \mathfrak{A}^*$: for each $a \in E$, $f(a) \in \mathfrak{A}(a)$, thus, $df(a) \in g(a)D$. The rational function df/g is then integer-valued at each a which is not a root of g. Since E is a d-set, it follows that g divides df in $K[X]$, that is, f belongs to the intersection $gK[X] \cap \mathrm{Int}(E,D)$. □

Remark 3.4. Assuming that $\mathrm{Int}(E,D)$ satisfies the almost Skolem property, we do not know if it is necessary that E be a d-set for $\mathrm{Int}(E,D)$ to satisfy the Skolem property, nor that if it is sufficient that E be an s-set. At least, we can show that it is sufficient that E be an almost d-set (each integer-valued rational function is a polynomial). Suppose indeed that $\mathfrak{A}$ is a finitely generated non-unitary ideal, then $\mathfrak{A} \subseteq \big(qK[X] \cap \mathrm{Int}(E,D)\big)$, for some polynomial q which is irreducible in $K[X]$. There is a nonzero element $d \in D$ such that $d\mathfrak{A} \subseteq q\mathrm{Int}(E,D)$. If $f \in \mathfrak{A}^*$, then $df(a) \in q(a)D$ for each $a \in E$.
— If q has no root in E, then df/q is an integer-valued rational function on E. Assuming that E is an almost d-set, it follows that df/q is a polynomial, that is, $f \in \big(qK[X] \cap \mathrm{Int}(E,D)\big)$.
— If $q(a) = 0$, then $q = X - a$ (since q is irreducible in $K[X]$). On the other hand, $f(a) = 0$ (since $df(a) \in q(a)D$). Hence $q = X - a$ divides f in $K[X]$, and again $f \in \big(qK[X] \cap \mathrm{Int}(E,D)\big)$.

Almost Skolem property. We just saw that, if E is a d-set, we may restrict ourselves to the almost Skolem property. Similarly to the case of $\mathrm{Int}(D)$ [Proposition 1.5], we can conclude in the case of a one-dimensional Noetherian domain with finite residue fields:

Proposition 3.5. *Let D be a one-dimensional Noetherian domain with finite residue fields, and E be a fractional subset of D. Then* $\mathrm{Int}(E,D)$ *satisfies the almost Skolem property.*

Proof. Replacing E by an isomorphic subset, one may in fact assume that E is a subset of D [Proposition 2.15]. We then know that the non-unitary maximal ideals of $\mathrm{Int}(E,D)$ are of the form $\mathfrak{M}_{\mathfrak{m},\alpha} = \{f \in \mathrm{Int}(E,D) \mid f(\alpha) \in \mathfrak{m}\widehat{D_\mathfrak{m}}\}$, where $\mathfrak{m}$ is a maximal ideal of D and α is an element of the topological closure of E in the $\mathfrak{m}$-adic completion of D [4, Proposition V.2.2] . Let $\mathfrak{A}$ be a proper unitary finitely generated ideal of $\mathrm{Int}(E,D)$. Since $\mathfrak{A}$ is a proper unitary ideal, it is contained in some maximal ideal $\mathfrak{M}_{\mathfrak{m},\alpha}$. Since $\mathfrak{A}$ is finitely generated, there is some $a \in E$ close enough to α such that $\mathfrak{A} \subseteq \mathfrak{M}_{\mathfrak{m},a}$, that is, such that $\mathfrak{A}(a) \neq D$.

Corollary 3.6. *Let D be an order of a number field and E be a fractional subset of D. If E is a d-set, then* $\mathrm{Int}(E,D)$ *satisfies the Skolem property.*

The results of the last section will show that we may also obtain a positive conclusion, in the local case, even if the residue field of D is infinite, under some compactness condition for E (and other conditions).

Skolem closure of divisorial ideals. For the unitary divisorial ideals, Lemma 1.3 may be extended without difficulties to a subset.

Lemma 3.7. *Each unitary divisorial ideal of* $\mathrm{Int}(E,D)$ *is Skolem closed.*

Proof. Let $\mathfrak{A}$ be a unitary divisorial ideal of $\mathrm{Int}(E,D)$. We first show that $\mathfrak{A}$ is an intersection of fractional principal ideals of the form $(1/q)\mathrm{Int}(E,D)$ where $q \in K[X]$. Let $\varphi\mathrm{Int}(E,D)$ be a principal ideal containing $\mathfrak{A}$ and write $\varphi = p/q$ where p, q are relatively prime in $K[X]$. Since $\mathfrak{A}$ is unitary, there is a nonzero element a in $\mathfrak{A} \cap D$. Since $a \in \mathfrak{A}$, a fortiori $a \in \varphi\mathrm{Int}(E,D)$: we may write $aq = ph$, where $h \in \mathrm{Int}(E,D)$. Since p and q are relatively prime, it follows that p is a constant, as claimed.

Now let $f \in \mathfrak{A}^*$ and let $q \in K[X]$ be such that $\mathfrak{A} \subseteq (1/q)\mathrm{Int}(E,D)$. For each $a \in E$, one has $f(a) \in \mathfrak{A}(a)$, and thus, $f(a)q(a) \in D$, that is, $fq \in \mathrm{Int}(E,D)$. Therefore, $f \in (1/q)\mathrm{Int}(E,D)$, and finally, f belongs to the ideal $\mathfrak{A}$ which is the intersection of the principal ideals that contain it. □

However, as noted in introduction of the notion of d-set, if the (non-unitary) principal ideals of $\mathrm{Int}(E,D)$ are Skolem closed, then E is a d-set. We show here that E must also be coherent. In fact, we have an equivalence.

Proposition 3.8. *Let E be a subset of K. The following assertions are equivalent*:

(i) *E is a coherent d-set,*
(ii) *the principal ideals of* $\mathrm{Int}(E,D)$ *are Skolem closed,*
(iii) *the divisorial ideals of* $\mathrm{Int}(E,D)$ *are Skolem closed.*

Proof. (i) ⇒ (iii) Let $\mathfrak{A}$ be a divisorial ideal of $\mathrm{Int}(E,D)$, that is, an intersection of fractional principal ideals. Consider $f \in \mathfrak{A}^*$. For each $a \in E$ we have $f(a) \in \mathfrak{A}(a)$. If $\varphi \in K(X)$ is such that the principal ideal $\varphi\mathrm{Int}(E,D)$ contains $\mathfrak{A}$, and if a is not a pole of φ, we then have $f(a) \in \varphi(a)D$. Hence, the rational function f/φ is almost integer-valued on E. Assuming that E is a coherent d-set, it follows that f/φ is in fact an integer-valued polynomial [Proposition 2.9], that is, $(f/\varphi) \in \mathrm{Int}(E,D)$. Hence $f \in \varphi\mathrm{Int}(E,D)$, and this holds for each principal ideal containing $\mathfrak{A}$.
(iii) ⇒ (ii) Obvious.
(ii) ⇒ (i) Let $\varphi = f/g$ be an almost integer-valued rational function, that is $\varphi(a) \in D$ for each $a \in E$, but possibly $\{a_1, \ldots, a_r\}$. Consider the polynomial $\psi = \prod_{i=1}^{r}(X - a_i)$. Then $(\psi f)(a) \in (\psi g)(a)D$ for each $a \in E$ (now, with no exception). Since the principal ideal $(\psi g)\mathrm{Int}(E,D)$ is Skolem closed, it follows that $\psi f = \psi g h$, where $h \in \mathrm{Int}(E,D)$. In conclusion, $\varphi = f/g = h$ is an integer-valued polynomial. □

Strong coherence and strong Skolem property. Analogously to Proposition 1.2, we can now relate the strong and almost strong Skolem properties.

Proposition 3.9. *Let E be a subset of K. Then* $\mathrm{Int}(E, D)$ *satisfies the strong Skolem property if and only if the following two conditions are satisfied*:

(a) *E is a strongly coherent d-set with respect to D,*
(b) $\mathrm{Int}(E, D)$ *satisfies the almost strong Skolem property.*

Proof. — The conditions are necessary. It is immediate that $\mathrm{Int}(E, D)$ must satisfy the almost strong Skolem property. From Proposition 3.8, we know also that E is a coherent d-set, but prove that, in fact, E is strongly coherent. Let $\mathfrak{A}$ be a finitely generated ideal of $\mathrm{Int}(E, D)$, and $f \in \mathrm{Int}(E, D)$ be such that $f(a) \in \mathfrak{A}(a)$ for almost each $a \in E$, that is, each a but possibly $\{a_1, \ldots, a_r\}$. As in the proof of Proposition 3.8, consider the polynomial $\psi = \prod_{i=1}^{r}(X - a_i)$, then set $\mathfrak{B} = \psi\mathfrak{A}$. Thus $(\psi f)(a) \in \mathfrak{B}(a)$ for each $a \in E$ (now, with no exception). From the strong Skolem property, we have $\psi f \in \mathfrak{B}$, that is, $\psi f = \psi h$, where $h \in \mathfrak{A}$. Thus $f \in \mathfrak{A}$, and of course, $f(a) \in \mathfrak{A}(a)$ for all $a \in E$.
— The conditions are sufficient. Let $\mathfrak{A}$ be a finitely generated nonzero ideal of $\mathrm{Int}(E, D)$, and f be in the Skolem closure $\mathfrak{A}^*$ of $\mathfrak{A}$: for each $a \in E$, $f(a) \in \mathfrak{A}(a)$. As for $\mathrm{Int}(D)$, we may find a polynomial $g \in D[X]$, and a nonzero element $d \in D$, such that $d\mathfrak{A} = g\mathfrak{B}$, where $\mathfrak{B}$ is a finitely generated unitary ideal of $\mathrm{Int}(E, D)$ [4, Lemma VI.1.2]. We thus have $df(a) \in g(a)\mathfrak{B}(a)$ for each $a \in E$. In particular, $\varphi = df/g$ is an almost integer-valued rational function. Since E is a coherent d-set, φ is in fact an integer-valued polynomial [Proposition 2.9]. By hypothesis $\varphi(a) \in \mathfrak{B}(a)$ for each $a \in E$, but possibly the zeros of g. Since φ is a polynomial and since E is strongly coherent, it follows that $\varphi(a) \in \mathfrak{B}(a)$ for all $a \in E$. Since $\mathfrak{B}$ is a unitary ideal and since $\mathrm{Int}(E, D)$ satisfies the almost strong Skolem property, it follows that $\varphi \in \mathfrak{B}$. Finally, $f = (g/d)\varphi$ belongs to $\mathfrak{A} = (g/d)\mathfrak{B}$. □

Prüfer domains. If $\mathrm{Int}(E, D)$ is a Prüfer domain, each finitely generated ideal is invertible, and a fortiori, divisorial. Hence it follows from Lemma 3.8 that it satisfies the strong Skolem property if and only if E is a coherent d-set. On the other hand, $\mathrm{Int}(E, D)$ satisfies always the almost strong Skolem property, and it follows from Proposition 3.9, that $\mathrm{Int}(E, D)$ satisfies the strong Skolem property if and only if E is a strongly coherent d-set. We thus obtain a partial generalization of Proposition 1.4, recovering also [12, Theorem 4.6]. This applies for instance to a fractional subset E of a Dedekind domain D with finite residue fields (we may replace E by a subset of D with the same properties [Proposition 2.15], then $\mathrm{Int}(E, D)$ is an overring of $\mathrm{Int}(D)$, which is a Prüfer domain).

Proposition 3.10. *Assume that* $\mathrm{Int}(E, D)$ *is a Prüfer domain* (*for instance, E is a fractional subset of a Dedekind domain with finite residue fields*), *then* $\mathrm{Int}(E, D)$ *satisfies the almost strong Skolem property. Moreover, the following assertions are equivalent*:

(i) *E is a coherent d-set,*
(ii) *E is a strongly coherent d-set,*
(iii) $\mathrm{Int}(E, D)$ *satisfies the strong Skolem property.*

However, if $\mathrm{Int}(E, D)$ satisfies the Skolem property, we know only that E is an s-set. We do not know if we can conclude that E is a strongly coherent d-set, we do not know if we can conclude that $\mathrm{Int}(E, D)$ satisfies the strong Skolem property (as in Proposition 1.4).

Example 3.11. Let E be a subset of $\mathbb{Z}$. Then E is a d-set if and only if it is infinite [Example 2.4 (2)]. On the other hand, since $\mathbb{Z}$ is completely integrally closed, E is coherent if and only if it has no polynomially isolated point [4, Exercise VI.10]. Since $\mathrm{Int}(\mathbb{Z})$ is a Prüfer domain, we conclude that $\mathrm{Int}(E, \mathbb{Z})$ satisfies the strong Skolem property, if and only if E is infinite without isolated points [12, Corollary 4.8].

4. Almost strong Skolem property: the local case

We do not know if the almost strong Skolem property for $\mathrm{Int}(E, D)$ implies that, for each maximal ideal $\mathfrak{m}$ of D, $\mathrm{Int}(E, D_{\mathfrak{m}})$ satisfies also this property (that is, whether Lemma 1.8 may be extended). Nevertheless, if $\mathrm{Int}(E, D_{\mathfrak{m}})$ satisfies the almost strong Skolem property for each maximal ideal $\mathfrak{m}$ of D, then clearly so does $\mathrm{Int}(E, D)$. Therefore, we restrict ourselves to the local case.

Notations and hypotheses. We let D be a one-dimensional local Noetherian domain with maximal ideal $\mathfrak{m}$, and E be a fractional subset of D. We denote by $\widehat{D}$ the completion of D, and by $\widehat{E}$ the topological closure of E (in the $\mathfrak{m}$-adic topology). We assume that D is analytically irreducible.

Assuming moreover that the residue field of D is finite, we have seen that $\mathrm{Int}(D)$ satisfies the almost strong Skolem property [Theorem 1.9]. This generalizes easily to a fractional subset E of D [4, Exercise VII.17]. The proof is in every respect similar to the case of $\mathrm{Int}(D)$, using the fact that $\widehat{E}$ is compact. However $\widehat{E}$ may be compact even if $D/\mathfrak{m}$ is infinite. This is clearly the case if E is finite (but we already know that $\mathrm{Int}(E, D)$ satisfies the almost super Skolem property in this case [Proposition 3.1]). Here are two less trivial examples:

Examples 4.1. 1) Let $k = \mathbb{F}_q$ be a finite field. The ring $D = \widehat{D} = k(y)[[t]]$ of power series with coefficients in $k(y)$ is not compact (its residue field is infinite); the subring $E = \widehat{E} = k[[t]]$ is compact (its residue field is finite).
2) Let $D = \widehat{D} = K[[t]]$ be the ring of power series with coefficients in an infinite field K. As above, D is not compact. Let $E = \{t^k \mid k \in \mathbb{N}^*\}$. Then $\widehat{E} = E \cup \{0\}$ is compact.

We shall prove that $\mathrm{Int}(E, D)$ satisfies the almost strong Skolem property, under the hypothesis that $\widehat{E}$ is compact (without supposing that the residue field of D is finite). We will most often assume that E is a subset of D (if E is a fractional subset, we may as well replace it by an isomorphic subset, of the form dE, which is contained in D). As for $\mathrm{Int}(D)$, the proof relies mainly on three points:
— A. The ring $\mathcal{C}(\widehat{E}, \widehat{D})$ of continuous functions from $\widehat{E}$ to $\widehat{D}$ satisfies the almost super Skolem property.
— B. Analogously to the classical Stone-Weierstrass theorem, the ring $\mathrm{Int}(E, D)$ is dense in the ring $\mathcal{C}(\widehat{E}, \widehat{D})$ for the uniform convergence topology [4, Corollary III.5.6].
— C. For each finitely generated unitary ideal $\mathfrak{A}$ of $\mathrm{Int}(E, D)$, the ideals of values are locally constant on E: there exists a nonzero ideal $\mathfrak{b}$ of D such that $(a - b) \in \mathfrak{b} \cap E$ implies $\mathfrak{A}(a) = \mathfrak{A}(b)$. It follows that $f(a) \in \mathfrak{A}(a)$ for each $a \in E$ implies that $f(x) \in \mathfrak{A}(x)\widehat{D}$ for each $x \in \widehat{E}$.

Assertion A: continuous functions. We first state and prove a very general result: the super Skolem property for the ring of continuous functions holds under the assumption that $\widehat{E}$ is compact.

Proposition 4.2. *Let X be a topological space and R be a one-dimensional local Noetherian domain, with maximal ideal $\mathfrak{m}$. Assume that X is compact and totally disconnected. Then $\mathcal{C}(X,R)$, the ring of continuous functions from X to R (when R is endowed with the $\mathfrak{m}$-adic topology), satisfies the almost super Skolem property.*

Proof. Let $\mathfrak{A}$ be a unitary ideal of $\mathcal{C}(X,R)$: there is a nonzero element a in $\mathfrak{A}\cap R$. Since R is a one-dimensional local Noetherian domain, there is an integer k such that $\mathfrak{m}^k \subseteq aR$. Consider $\varphi \in \mathfrak{A}^*$, we wish to prove that $\varphi \in \mathfrak{A}$. By hypothesis, for each $x \in X$, there is a function $\psi_x \in \mathfrak{A}$ such that $\varphi(x) = \psi_x(x)$. By continuity of the functions φ and ψ_x, there is a neighborhood U_x of x such that, for each $y \in U_x$, $\big(\varphi(y) - \psi_x(y)\big) \in \mathfrak{m}^k R$, and thus $\big(\varphi(y) - \psi_x(y)\big) \in aR$. Since X is totally disconnected, we may choose each U_x to be a clopen subset of X. Since X is compact, it may be covered by finitely many such $U_1, \ldots, U_s$, with corresponding functions $\psi_1, \ldots, \psi_s$, such that, for each $y \in U_j$, $\big(\varphi(y) - \psi_j(y)\big) \in aR$. Since these sets are clopen sets, we may even choose them to be pairwise disjoint. Let η_j be the characteristic function of U_j, then η_j is continuous. Set $\psi = \sum_{i=1}^s \psi_i\eta_i$, then $\psi \in \mathfrak{A}$. For each $y \in X$, there is j such that $y \in U_j$, and hence,

$$\varphi(y) - \psi(y) = \varphi(y) - \sum_{i=1}^{s} \psi_i(y)\eta_i(y) = \varphi(y) - \psi_j(y).$$

Therefore $\big(\varphi(y) - \psi(y)\big) \in aR$, that is, $(1/a)\big(\psi(y) - \varphi(y)\big) \in R$. In other words, the function $\theta = (1/a)(\psi - \varphi)$ belongs to $\mathcal{C}(X,R)$. Finally, since $\psi \in \mathfrak{A}$ and $a \in \mathfrak{A}$, we have $\varphi = \psi - a\theta \in \mathfrak{A}$. $\square$

For instance, $R = \widehat{D}$ may be the completion of an analytically irreducible domain D and $X = \widehat{E}$ be the topological closure of a subset E of D. Assuming that $\widehat{E}$ is compact, we conclude that the ring $\mathcal{C}(\widehat{E},\widehat{D})$ of continuous functions from $\widehat{E}$ to $\widehat{D}$ satisfies the almost super Skolem property.

Assertion B: Stone-Weierstrass.

Proposition 4.3. *Let D be a one-dimensional local Noetherian analytically irreducible domain (with maximal ideal $\mathfrak{m}$), and E be a subset of D such that the completion $\widehat{E}$ of E (in the $\mathfrak{m}$-adic topology) is compact. Then $\mathrm{Int}(E,D)$ is dense in $\mathcal{C}(\widehat{E},\widehat{D})$ for the uniform convergence topology.*

Proof. We first consider the case where D is a rank-one discrete valuation domain. We denote by K the quotient field of D and by $\widehat{K}$ its completion (in the topology given by the corresponding valuation, which clearly coincides with the $\mathfrak{m}$-adic topology on D). The polynomials with coefficients in $\widehat{K}$ can be considered as (uniformly) continuous functions from $\widehat{E}$ to $\widehat{K}$, and it follows from a classical generalization of the Stone-Weierstrass theorem, that $\widehat{K}[X]$ is dense in $\mathcal{C}(\widehat{E},\widehat{K})$ for the uniform convergence topology ($\widehat{E}$ is compact and $\widehat{K}[X]$ “separates the points” of $\widehat{E}$: for $a \neq b$ in $\widehat{E}$, the polynomial $f = \frac{X-a}{b-a}$ is such that $f(a) = 0$, and $f(b) = 1$) [15], [4, Exercise III.20]. Clearly $K[X]$ is dense in $\widehat{K}[X]$, and hence, in $\mathcal{C}(\widehat{E},\widehat{K})$. Finally, $\mathcal{C}(\widehat{E},\widehat{D})$ is an open set in $\mathcal{C}(\widehat{E},\widehat{K})$. Hence the intersection of $K[X]$ with this open set, that is, $\mathrm{Int}(E,D) = K[X]\cap\mathcal{C}(\widehat{E},\widehat{D})$, is dense in $\mathcal{C}(\widehat{E},\widehat{D})$.

Now, we consider the general case: D is a one-dimensional local Noetherian analytically irreducible domain. The integral closure D' of D is a rank-one discrete valuation domain, with maximal ideal $\mathfrak{m}'$, moreover D' is a finitely generated D-module [21, (32.2)]. Therefore, the $\mathfrak{m}'$-adic topology on D' induces the $\mathfrak{m}$-adic topology on D; in particular, there is an integer k such that $\mathfrak{m}'^k \subseteq \mathfrak{m}$. As in [4, Proposition III.2.4], it is not difficult to see that $\mathrm{Int}(E,D)$ is dense in $\mathcal{C}(\widehat{E},\widehat{D})$, provided that, for each h and each n, the characteristic function of $\mathfrak{m}^h\cap E$ can be approximated modulo $\mathfrak{m}^n$ by an integer-valued polynomial (or equivalently the characteristic function of $\widehat{\mathfrak{m}}^h\cap\widehat{E}$ can be approximated modulo $\widehat{\mathfrak{m}}^n$ by an integer-valued polynomial). Clearly $\mathfrak{m}^h\cap E$ is a clopen set of E in the $\mathfrak{m}$-adic topology, hence also in the $\mathfrak{m}'$-adic topology (and $\widehat{\mathfrak{m}}^h\cap\widehat{E}$ is a clopen set of $\widehat{E}$ in the $\widehat{\mathfrak{m}}'$-adic topology). From the special case above of a valuation domain, there is a polynomial $f\in\mathrm{Int}(E,D')$ such that

$$f(z)\equiv\begin{cases} 1 \pmod{\mathfrak{m}'^{kn}}, & \text{if } z\in\mathfrak{m}^h\cap E,\\ 0 \pmod{\mathfrak{m}'^{kn}}, & \text{if } z\in E, z\notin\mathfrak{m}^h.\end{cases}$$

Since $\mathfrak{m}'^{kn}\subseteq\mathfrak{m}^n\subset D$, one has $f(E)\subseteq D$, and hence f is an approximation in $\mathrm{Int}(E,D)$ of the characteristic function of $\mathfrak{m}^h\cap E$ modulo $\mathfrak{m}^n$. □

Compactness: a necessary condition. We obtained positive results under the assumption that $\widehat{E}$ is compact. It follows from the next two lemmas that this condition is necessary. The first one is a classical topological argument, we give it for the sake of completeness.

Lemma 4.4. *Let E be a subset of a local domain R, with maximal ideal $\mathfrak{m}$. Suppose that $\bigcap_n\mathfrak{m}^n=(0)$ and denote by $\widehat{E}$ the topological closure of E in the completion of R in the $\mathfrak{m}$-adic topology. Then $\widehat{E}$ is compact if and only if E meets only finitely many cosets of R modulo $\mathfrak{m}^n$ for each n.*

Proof. That the condition is necessary is clear. Conversely, since R is a metric space, to show that $\widehat{E}$ is compact amounts to show that, from a sequence $\{x_n\}$ in $\widehat{E}$, we may extract a converging subsequence. Infinitely many terms of the sequence $\{x_n\}$, forming a subset X_1 of $\widehat{E}$, are in the same coset modulo $\mathfrak{m}$ (since E meets only finitely much such cosets). Then infinitely many terms of X_1, forming a subset X_2 of X_1, are in the same coset modulo $\mathfrak{m}^2$. And so on. We thus define a decreasing sequence $\{X_n\}$ of subsets, the elements of X_n being in the same coset modulo $\mathfrak{m}^n$. Let x_{k_n} be the first term of X_n. The subsequence $\{x_{k_n}\}$ of $\{x_n\}$ is a Cauchy sequence in $\widehat{E}$, and hence, it converges. □

Lemma 4.5. *Let D be a one-dimensional local Noetherian analytically irreducible domain, with maximal ideal $\mathfrak{m}$, and E be a subset of D. If $\mathrm{Int}(E,D)$ satisfies the almost strong Skolem property, then E meets only finitely many cosets of D modulo $\mathfrak{m}^n$ for each n.*

Proof. Assuming that E meets infinitely many cosets of D modulo some $\mathfrak{m}^k$, we shall prove that $\mathrm{Int}(E,D)$ does not satisfy the almost strong Skolem property.

— The integral closure D' of D is a discrete valuation domain with maximal ideal $\mathfrak{m}'$ and there is an integer r such that $\mathfrak{m}'^r \subseteq \mathfrak{m}$. Hence, E meets infinitely many cosets of D' modulo $\mathfrak{m}'^{rk}$.
— If n is the greatest integer such that E meets only finitely many cosets of D' modulo $\mathfrak{m}'^n$, there is a coset $a+\mathfrak{m}'^n$ such that $F = E \cap (a+\mathfrak{m}'^n)$ meets infinitely many cosets of D' modulo $\mathfrak{m}'^{n+1}$. Denoting by t a generator of the ideal $\mathfrak{m}'$, we may replace E and F by the subsets $E' = -a + (1/t^n)E$ and $F' = -a + (1/t^n)F$, which have the same Skolem properties [Proposition 2.15]. Since E contains F, it follows that E' contains F'. But now, F' is a subset of D' which meets infinitely many cosets of D' modulo $\mathfrak{m}'$. From [4, Proposition I.3.1], we then have the containments

$$\mathrm{Int}(E', D) \subseteq \mathrm{Int}(F', D) \subseteq \mathrm{Int}(F', D') \subseteq D'[X].$$

— Consider the ideal $\mathfrak{A}$ of $\mathrm{Int}(E,D)$ generated by the maximal ideal $\mathfrak{m}$ of D and X^2. It is easy to see that X belongs to the Skolem closure $\mathfrak{A}^*$ of $\mathfrak{A}$. By way of contradiction, assume that $\mathrm{Int}(E,D)$ satisfies the almost strong Skolem property. Then $X \in \mathfrak{A}$, and a fortiori, $X \in (\mathfrak{m}', X^2)D'[X]$. We reach a contradiction. □

Assertion C and conclusion. Lastly, we need to prove assertion C: the value ideals of a finitely generated unitary ideal of $\mathrm{Int}(E,D)$ are locally constant on E. This is very similar to the case of $\mathrm{Int}(D)$ [4, Lemma VII.1.9].

Lemma 4.6. *Let D be a domain, E be a subset of D, and $\mathfrak{A}$ be a finitely generated unitary ideal of* $\mathrm{Int}(E,D)$. *Then there exists a nonzero ideal $\mathfrak{b}$ of D such that $(a-b) \in \mathfrak{b} \cap E$ implies $\mathfrak{A}(a) = \mathfrak{A}(b)$.*

Proof. Let $f_1, \ldots, f_r$ be a system of generators of $\mathfrak{A}$, and d be a nonzero common denominator of their coefficients. For each a, b in D, and for each i, $(a-b)$ divides $\big(df_i(a) - df_i(b)\big)$ [4, Lemma I.3.19]. In particular, if $\mathfrak{a} = \mathfrak{A} \cap D$, and $\mathfrak{b} = d\mathfrak{a}$, then $(a-b) \in \mathfrak{b}$ implies $\big(f_i(a) - f_i(b)\big) \in \mathfrak{a}$. On the other hand, if a and b are in E, then $\mathfrak{A}(a)$ and $\mathfrak{A}(b)$ contain $\mathfrak{a}$. Therefore $(a-b) \in \mathfrak{b} \cap E$ implies

$$\mathfrak{A}(a) = \big(\mathfrak{a}, f_1(a), \ldots, f_r(a)\big) = \big(\mathfrak{a}, f_1(b), \ldots, f_r(b)\big) = \mathfrak{A}(b).$$

□

Putting together all the results of this section, we can finally characterize the fractional subsets E of D such that $\mathrm{Int}(E,D)$ satisfies the almost strong Skolem property.

Theorem 4.7. *Let D be a one-dimensional local Noetherian analytically irreducible domain (with maximal ideal $\mathfrak{m}$), and E be a fractional subset of D. Then* $\mathrm{Int}(E,D)$ *satisfies the almost strong Skolem property if and only if the completion $\widehat{E}$ of E (in the $\mathfrak{m}$-adic topology) is compact.*

Proof. From Lemma 4.4 and Lemma 4.5, we know that the compactness of $\widehat{E}$ is necessary. Let us prove that it is sufficient. Let $\mathfrak{A} = (f_1, \ldots, f_r)$ be a finitely generated unitary ideal of $\mathrm{Int}(E,D)$. Then $\mathfrak{A}$ contains a nonzero constant $a \in D$, and there is an integer k such that $\mathfrak{m}^k \subseteq aD$. Consider $f \in \mathfrak{A}^*$: for each $a \in D$, $f(a) \in \mathfrak{A}(a)$. From Lemma 4.6, it follows that, for each $x \in \widehat{E}$, $f(x) \in \mathfrak{A}(x)\widehat{D}$. From Proposition 4.2, we know that $\mathcal{C}(\widehat{E}, \widehat{D})$ satisfies the almost super Skolem property: f belongs to the ideal $\mathfrak{A}\mathcal{C}(\widehat{E}, \widehat{D})$ and we can write $f = \sum_{j=1}^r f_j \psi_j$, where each ψ_j belongs to $\mathcal{C}(\widehat{E}, \widehat{D})$. From the Stone-Weierstrass property [Proposition 4.3], each

ψ_j can be approximated by an integer-valued polynomial $g_j \in \text{Int}(E, D)$, modulo $\mathfrak{m}^k\widehat{D}$: for each $x \in E$, $g_j(x) - \psi_j(x) \in \mathfrak{m}^k\widehat{D}$. Let $g = \sum_{j=1}^r f_j g_j$, then $g \in \mathfrak{A}$. The difference $(f - g)$ is a polynomial, and, for each $x \in E$, we have

$$(f - g)(x) = \sum_{j=1}^{r} f_j(x)(\psi_j(x) - g_j(x)) \in \mathfrak{m}^k\widehat{D}.$$

It follows that $(f - g)(x) \in aD$, that is, $(f - g) = ah$, where $h \in \text{Int}(E, D)$. Finally, $f = g + ah$ belongs to the ideal $\mathfrak{A}$. □

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