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VALUATIVE DOMAINS

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Abstract. A (commutative integral) domain R is said to be valuative if, for each nonzero element u in the quotient field of R, at least one of $R \subseteq R[u]$ and $R \subseteq R[u^{-1}]$ has no proper intermediate rings. Such domains are closely related to valuation domains. If R is a valuative domain, then R has at most three maximal ideals, and at most two if R is not integrally closed. Also, if R is valuative, the set of nonmaximal prime ideals of R is linearly ordered, at most one maximal ideal of R does not contain each nonmaximal prime of R, and R_P is a valuation domain for each prime P except for at most one maximal ideal. Any integrally closed valuative domain is a Bézout domain. Valuation domains are characterized as the quasilocal integrally closed valuative domains. Each one-dimensional Prüfer domain with at most three maximal ideals is valuative.

1. Introduction

Throughout the paper, R denotes a (commutative integral) domain with quotient field K, and R' denotes the integral closure of R (in K). As usual, the set of prime (resp., maximal) ideals of R is denoted by Spec(R) (resp., Max(R)); the (Krull) dimension of R is denoted by dim(R); an *overring of* R is any ring T such that $R \subseteq T \subseteq K$; $\sqrt{I}$ denotes the radical of an ideal I; and $|S|$ denotes the cardinality of a set S. Any unexplained material is standard, as in [14], [17].

One of the characterizing features of a valuation domain R is that either u or u^{-1} is in R for each nonzero element u of K. Relaxing this condition, we say that R is *valuative* if either $R \subseteq R[u]$ or $R \subseteq R[u^{-1}]$ has no proper intermediate rings. In the event that neither u nor u^{-1} is in R, then at least one of the two pairs must be a minimal extension (in the sense of [11], [9]). Our purpose here is to study the class

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of valuative domains. This study naturally considers separately the cases in which such a domain is or is not integrally closed, for if $D \subsetneq E$ is a minimal extension, then either D is integrally closed in E (a situation that we termed a *closed minimal extension* in [4]) or E is integral over D (a situation that we termed a *minimal integral extension*).

In Section 2 we first establish a few general necessary conditions on the prime ideals of a valuative domain. We show that a valuative domain has at most three maximal ideals, the set of its nonmaximal primes is linearly ordered and at most one maximal ideal does not contain each nonmaximal prime (Theorem 2.2). We also prove that a valuative domain is locally valuative (Proposition 2.4). The converse is obviously false: a Prüfer domain is locally valuative but may have more than three maximal ideals (the simplest example being the ring of integers!) or may have two or three maximal ideals with two of them failing to contain every nonmaximal prime (see figures (5) and (6) following Corollary 3.11). At the end of the paper, we even produce an example (in the non-integrally closed case) of a locally valuative domain R with two maximal ideals, both containing each nonmaximal prime, but such that R is not valuative (Example 6.8).

In the next section we characterize the integrally closed valuative domains as the Bézout domains with at most three maximal ideals such that at most one maximal ideal does not contain each nonmaximal prime (Theorem 3.7). We derive that the integral closure of a valuative domain is valuative (Corollary 3.10). For finite-dimensional Bézout domains, the size of $\mathrm{Spec}(R)$ almost determines whether R is valuative or not. In fact, if R is an n-dimensional Bézout domain, then R is valuative if and only if $|\,\mathrm{Spec}(R)| \leq n+3$ such that if $|\,\mathrm{Spec}(R)| = n+3$, then R has three maximal ideals and at least two of these maximal ideals are of height n (Corollary 3.11).

Turning our attention to the non-integrally closed case, we note that, for each element u in the integral closure R' of a valuative domain R, either $u \in R$ or $R[u]$ is a minimal extension of R (Proposition 4.1). We thus say a ring extension $R \subsetneq T$ is a *pointwise minimal extension* if for each $t \in T \backslash R$, $R \subsetneq R[t]$ is a minimal extension, and we devote Section 4 to this notion. If $R \subsetneq T$ is a pointwise minimal overring extension of domains, we show that there exists a maximal ideal M of R, which we call the *crucial maximal ideal* of the extension, such that, as for minimal extensions, $R_N = T_N := T_{R\setminus N}$ for each maximal ideal N of R that is distinct from M and $R_M \subsetneq T_M$ is a pointwise minimal extension (Theorem 4.5). In the case of a pointwise minimal integral overring extension, we show moreover that $M = (R : T)$ is an ideal of T (Theorem 4.6). This allows in particular to view a valuative domain as obtained from a valuative Bézout domain via a pullback construction. We derive that a valuative domain R is such that R_P is a valuation domain for each prime P except for possibly one maximal ideal M (Corollary 4.8).

Section 5 is devoted to quasilocal valuative domains that are not necessarily integrally closed. Proposition 5.1 gives a general characterization: R is (quasilocal and) valuative if and only if R' is a valuation domain and either $R = R'$ or $R \subsetneq R'$ is a pointwise minimal extension. We derive that the maximal ideal M of R must be an ideal of R'. We then split the characterization of the quasilocal valuative domains in two cases, determined by whether or not M is the maximal ideal of R'. Recall from [16] that a pseudo-valuation domain is a quasilocal domain R whose maximal ideal M is also the maximal ideal of a valuation domain V that (necessarily) contains R. We characterize the valuative pseudo-valuation domains in Theorem 5.2 and deal with the case where R is not a pseudo-valuation domain in Theorem 5.10. For pseudo-valuation domains, the characterization is simply given by a condition on the field extension $R/M \subseteq V/M$. It follows, for instance, that $\mathbb{Q}+X\mathbb{Q}(\sqrt{2})[[X]]$ is a valuative pseudo-valuation domain and that $\mathbb{Q}+X\mathbb{Q}(\sqrt[4]{2})[[X]]$ is a pseudo-valuation domain that is not valuative. For (quasilocal) domains that are not pseudo-valuation domains, together with similar conditions on the field extension $R/M \subsetneq R'/N$ (where N is the maximal ideal of R', distinct from M in this case), the valuation domain R' must be such that its maximal ideal N is principal, and, moreover M is not too far from the maximal ideal N as one must have $M = N^2$.

In the last section, we show that a non-integrally closed valuative domain has at most two maximal ideals. Theorem 6.2 gives a characterization of valuative domains that are neither quasilocal nor integrally closed. Specifically, if R is neither integrally closed nor quasilocal, then it is valuative if and only if it has exactly two maximal ideals N and M such that each nonmaximal prime is contained in M, R_N is a valuation domain, and R_M is a valuative pseudo-valuation domain that is not integrally closed. Moreover, R' has two maximal ideals, M (necessarily the conductor of R' into R) and N' (necessarily such that $N \subsetneq N'$ with $R_N = R'_{N'}$) (Corollary 6.3). In Example 6.7, we construct a (valuative) non-integrally closed domain R such that R' is the unique minimal integral overring of R and $R \subsetneq R_M$ is a closed minimal extension for the maximal ideal $M = (R' : R)$.

Throughout, we use the fact that if $R \subsetneq R[u]$ is a minimal ring extension, then $\sqrt{(R :_R u)} \in \mathrm{Max}(R)$. For a proof of this fact, see [11, Théorème 2.2 (ii)] (cf. also [4, Theorem 2.3]) for minimal integral extensions and [4, Theorem 3.4] for closed minimal extensions. Also, following [17, page 28], we let LO and GD denote the lying-over and going-down properties of ring extensions.

2. Necessary conditions

Prime spectrum. We first establish a necessary technical condition, and then infer more natural although rather unusual constraints.

Lemma 2.1. *If I and J are comaximal proper ideals of a valuative domain R, then at least one of $\sqrt{I}$ and $\sqrt{J}$ is a maximal ideal of R.*

Proof. We show the contrapositive: if I and J are comaximal proper ideals of a domain R such that neither $\sqrt{I}$ nor $\sqrt{J}$ is a maximal ideal of R, then R is not valuative. As I and J are comaximal, $aR + bR = R$ for some elements $a \in I$ and $b \in J$. Set $u := b/a$. For $t \in (R :_R u)$, $tb = as$ with $s \in R$ implies t belongs to each prime ideal containing a and, *a fortiori*, to each prime ideal containing I. Thus $\sqrt{(R :_R u)} \subseteq \sqrt{I}$; similarly, $\sqrt{(R :_R u^{-1})} \subseteq \sqrt{J}$. As neither $\sqrt{I}$ nor $\sqrt{J}$ is maximal, neither $R \subsetneq R[u]$ nor $R \subsetneq R[u^{-1}]$ is a minimal extension (although each is a proper extension). Thus R is not valuative. ☐

We derive three necessary conditions about the prime ideals of R.

Theorem 2.2. *Let R be a valuative domain. Then R satisfies the following three conditions.*

- (*i*) *R has at most three maximal ideals.*
- (*ii*) *The set of nonmaximal prime ideals of R is linearly ordered by inclusion.*
- (*iii*) *At most one maximal ideal does not contain each nonmaximal prime ideal.*

Proof. We prove each condition by way of contradiction.

i) Assume that a domain R has at least four maximal ideals M_1, M_2, M_3, M_4. Then $M_1M_2 + M_3M_4 = R$, as no maximal ideal of R can contain both M_1M_2 and M_3M_4. Note that neither M_1M_2 ($= M_1 \cap M_2$) nor M_3M_4 is a maximal ideal of R. By Lemma 2.1, R is not valuative.

ii) Assume there are incomparable nonmaximal primes P and Q. Let $u = c/b$ where $b \in P\backslash Q$ and $c \in Q\backslash P$. Then P contains the ideal $(R :_R u)$ while Q contains the ideal $(R :_R u^{-1})$. It follows that both $\sqrt{(R :_R u)}$ and $\sqrt{(R :_R u^{-1})}$ are proper nonmaximal ideals of R. Hence both $R \subsetneq R[u]$ and $R \subsetneq R[u^{-1}]$ are proper extensions of R but neither is a minimal extension. Thus R is not valuative.

iii) Assume two distinct maximal ideals M and N each fail to contain each nonmaximal prime. As the nonmaximal primes are linearly ordered, M and N both fail to contain some nonmaximal prime P. Thus P is comaximal with $M \cap N$, although neither P nor $M \cap N$ is maximal, contradicting Lemma 2.1. ☐

Remarks 2.3. 1) From (ii), it follows that a valuative domain is treed (i.e., no maximal ideal can contain incomparable primes). In a treed domain, condition (iii) implies condition (ii). Moreover, if some maximal ideal of a treed domain R contains each nonmaximal prime of R, then the nonmaximal primes of R are linearly ordered by inclusion.

2) If $|\operatorname{Max}(R)| \leq 2$, condition (ii) clearly implies condition (iii).

3) We shall see that the necessary conditions of Theorem 2.2 are sufficient in the case of a Prüfer domain [Theorem 3.7]. As a special case, any Dedekind domain with at most three maximal ideals is valuative.

Localization. Recall from [11, Lemme 1.3] that for a multiplicative subset S of R, if $R \subsetneq T$ is a minimal extension, then either $R_S = T_S$ canonically or $R_S \subsetneq T_S$ is a minimal extension. A simple consequence is that valuative domains are stable under localization.

Proposition 2.4. *If R is a valuative domain and S is a multiplicative subset of R (with $0 \notin S$), then R_S is valuative.*

In particular, a valuative domain R is such that R_P is valuative for each prime P of R. In fact, we will see that more is true: R_P is a valuation domain for each prime P except for at most one maximal ideal (Corollary 4.8). The converse obviously fails: a Prüfer domain with more than three maximal ideals is locally valuative but not valuative by Theorem 2.2 (i). In Example 6.8, we even give an example of a domain R with exactly two maximal ideals M and N such that both R_M and R_N are valuative (indeed R_N is a valuation domain) but R is not valuative. In that example, the only nonmaximal prime of R is (0) and so, trivially, both M and N contain each nonmaximal prime and the set of nonmaximal primes is linearly ordered.

3. Integrally closed valuative domains

Integral closure. We first look at the quasilocal case.

Proposition 3.1. *If* (R, M) *is a quasilocal valuative domain, then* R' *is a valuation domain.*

Proof. Let $0 \neq u \in K$. Without loss of generality, neither u nor u^{-1} is in R. Since R is valuative, we may assume that $R \subsetneq R[u]$ is a minimal extension. Then either $u \in R'$ or $R \subsetneq R[u]$ is a minimal closed extension. In the latter case, R quasilocal implies $u^{-1} \in M$, by the proof of (1) $\Rightarrow$ (2) in [9, Theorem 3.1], and *a fortiori* $u^{-1} \in R'$. Hence, in all cases, either u or u^{-1} is in R'. $\square$

Corollary 3.2. *Let* (R, M) *be a quasilocal integrally closed domain. Then* R *is valuative if and only if* R *is a valuation domain.*

Recall from [19] that a domain R is called an *i-domain* if the canonical map $\operatorname{Spec}(T) \to \operatorname{Spec}(R)$ is an injection for each overring T of R. By [19, Corollary 2.15, Proposition 2.34], a domain R is a quasilocal i-domain if and only if R' is a valuation domain, equivalently, if and only if each overring of R is quasilocal. As being an i-domain is clearly a local property of domains, Propositions 2.4 and 3.1 combine to yield the following result.

Corollary 3.3. *If* R *is a valuative domain, then* R *is an* i*-domain.*

Corollary 3.4. *Let* R *be a valuative domain* (*or, more generally, an* i*-domain*)*. If* S *is an overring of* R*, then the canonical map* $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ *induces an order-isomorphism of posets* (*that is, an order-preserving and order-reflecting bijection*) *from* $\operatorname{Spec}(S)$ *onto the set* $\{P \in \operatorname{Spec}(R) \mid PS \neq S\}$*. In particular,* $\operatorname{Spec}(R')$ *is order-isomorphic to* $\operatorname{Spec}(R)$*.*

Proof. By Corollary 3.3, a valuative domain is an i-domain. So we may assume that R is an i-domain. Then the map sending $Q \in \operatorname{Spec}(S)$ to $Q \cap R$ is injective. This map is also order-preserving. Moreover, $R \subseteq S$ satisfies GD, since any i-domain is a going-down domain in the sense of [6] (see [7, Corollary 2.3 or Corollary 2.5] or [19, Corollary 2.13]). For primes $P \subseteq M$ of R with $MS \neq S$, there is a maximal ideal N of S such that $M \subseteq N$. Then by GD, there are primes $P' \subseteq M' \subseteq N$ of S such that $P = P' \cap R$ and $M = M' \cap R$. By injectivity, P' is the only prime lying over P and M' is the only prime lying over M. Hence the contraction map is a natural order-isomorphism between the posets $\operatorname{Spec}(S)$ and $\{P \in \operatorname{Spec}(R) \mid PS \neq S\}$. That $\operatorname{Spec}(R)$ and $\operatorname{Spec}(R')$ are order-isomorphic follows from the fact that integral extensions satisfy LO. □

Corollary 3.5. *If R is a valuative domain, then R' is a Bézout domain with at most three maximal ideals.*

Proof. If $M \in \operatorname{Max}(R)$, then R_M is valuative, and so R'_M is a valuation domain. Thus (or by combining Corollary 3.3 and [19, Proposition 2.14]), it follows that R' is a Prüfer domain. By Corollary 3.4 and Theorem 2.2(i), R' has at most three maximal ideals. Finally, it is well known that a Prüfer domain with only finitely many maximal ideals must be a Bézout domain. □

Characterization of integrally closed valuative domains. We obtained above (Corollary 3.5) that an integrally closed valuative domain is a Bézout domain with at most three maximal ideals. In fact, the necessary conditions of Theorem 2.2 turn out to be sufficient for Bézout domains. First we show that the technical condition of Lemma 2.1 has a partial converse.

Lemma 3.6. *Let R be a Bézout domain. Then the following conditions are equivalent:*

1. *R is a valuative domain;*
2. *if I and J are comaximal proper ideals of R, then at least one of $\sqrt{I}$ and $\sqrt{J}$ is a maximal ideal of R.*

Proof. That (1) implies (2) is Lemma 2.1. For the converse, we let u be a nonzero element of K, and show that at least one of $R \subseteq R[u]$ and $R \subseteq R[u^{-1}]$ has no proper intermediate rings. Since R is Bézout, we can write $u = a/b$ with $aR + bR = R$.

If at least one of a, b is a unit of R, then either u or u^{-1} is in R and the assertion follows. Thus, we may assume that neither a nor b is a unit of R. By (2), it follows (possibly by interchanging a and b) that $M := \sqrt{bR} \in \operatorname{Max}(R)$. Then $u \in R_Q$ for each prime $Q \neq M$. Hence $R \subsetneq R[u] \subseteq \bigcap\{R_Q \mid Q \in \operatorname{Spec}(R) \setminus \{M\}\}$. As R is a Bézout domain, each overring is a ring of fractions of R [5, page 203]. Thus $R[u] = \bigcap\{R_Q \mid Q \in \operatorname{Spec}(R) \setminus \{M\}\}$ and there are no proper intermediate rings in the extension $R \subsetneq R[u]$. ☐

Theorem 3.7. *Let R be an integrally closed domain. Then the following conditions are equivalent:*

1. *R is a valuative domain;*
2. *R is a Bézout domain with at most three maximal ideals and at most one maximal ideal of R does not contain each nonmaximal prime ideal of R.*

Proof. (1) ⇒ (2): Combine Corollary 3.5 and Theorem 2.2.

(2) ⇒ (1): Assume (2). Without loss of generality, R is not quasilocal (since any valuation domain satisfies (1)). By Lemma 3.6, it is enough to show that if I and J are comaximal proper ideals of R, then at least one of $\sqrt{I}$ and $\sqrt{J}$ is a maximal ideal of R. Suppose that $\sqrt{I}$ is not maximal. Then either some nonmaximal prime ideal P contains I (which makes I a subset of all but one maximal ideal) or R has three maximal ideals with I contained in exactly two of them. It is straightforward to check that in either case, only one maximal ideal contains J and no nonmaximal prime contains J. Thus $\sqrt{J}$ is maximal. ☐

Remark 3.8. 1) In case R is a Bézout domain with at most two maximal ideals, we obtain a simpler sounding characterization: such an R is a valuative domain if and only if the set of nonmaximal prime ideals of R is linearly ordered under inclusion (cf. Remark 2.3.2).

2) The one-dimensional case is rather trivial: a one-dimensional integrally closed domain R is valuative if and only if R is a Bézout domain with at most three maximal ideals.

To motivate the next result, note that each overring of an i-domain is an i-domain.

Corollary 3.9. *Let R be an integrally closed valuative domain. Then each overring of R is a valuative Bézout domain.*

Proof. If R is an integrally closed valuative domain, then it is a Bézout domain, by Corollary 3.5. Hence each overring of R is a ring of fractions of R [5, page 203]. The assertion follows since valuative domains and Bézout domains are each stable under localization. $\square$

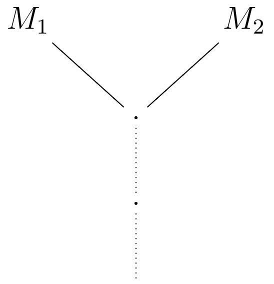
However we shall see that, in general, it is not always true that each overring of a valuative domain is valuative (Proposition 5.12). Yet, we finally also have the following.

Corollary 3.10. *If R is a valuative domain, then R' is a valuative Bézout domain.*

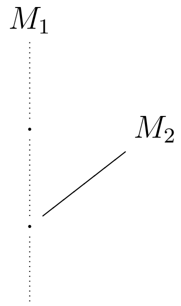
Proof. Assume R is valuative. By Corollary 3.5, R' is a Bézout domain with at most three maximal ideals. By Theorem 2.2, at most one maximal ideal of R does not contain each nonmaximal prime. The same property holds for R' since, by Corollary 3.4, there is an order-isomorphism between $\operatorname{Spec}(R)$ and $\operatorname{Spec}(R')$. Theorem 3.7 finally allows us to conclude that R' is valuative. $\square$

Pictures. For a valuative domain with exactly two maximal ideals, two cases may occur. We illustrate these by the following pictures (where broken lines represent arbitrary long linearly ordered chains of primes).

(1)

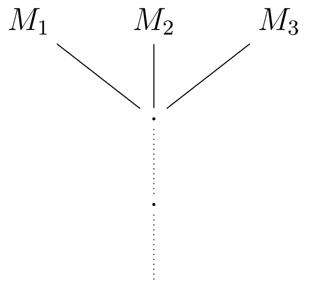


(2)

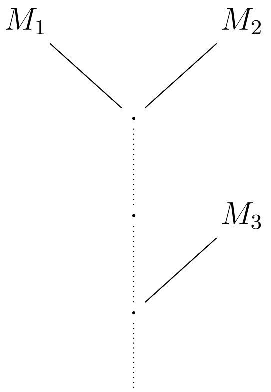


Similarly, there are two cases for a valuative domain with three maximal ideals.

(3)



(4)



Note that in case (2), there need not be a largest nonmaximal prime. In cases (2) and (4), at least one nonmaximal prime is not contained in some maximal ideal and there need not be a smallest such prime. Construction of Bézout domains with the above kinds of spectra is classical (cf. [18, Theorem 3.1]).

For a finite-dimensional Bézout domain R, we show next how $|\operatorname{Spec}(R)|$ can play a role in a characterization of the "valuative" property.

Corollary 3.11. *Let R be a finite-dimensional Bézout domain, with $n := \dim(R)$. Then:*

(a) *If R is valuative, then $|\operatorname{Spec}(R)| = n + |\operatorname{Max}(R)| \leq n + 3$.*

(b) *R is valuative if and only if either (i) $|\operatorname{Spec}(R)| \leq n+2$, or (ii) $|\operatorname{Spec}(R)| = n + 3$, R has three maximal ideals, and at least two of these maximal ideals have height n.*

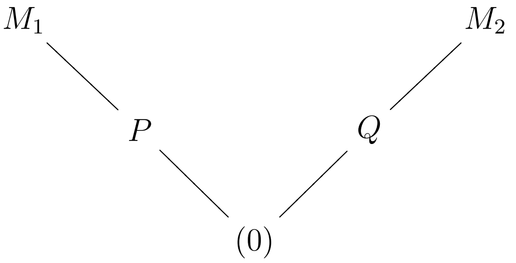
Proof. Observe that if D is any domain, then $\dim(D)$ is the maximal cardinality of a chain of nonmaximal primes of D, and so $\dim(D) = n$ implies $|\operatorname{Spec}(D)| \geq n + |\operatorname{Max}(D)|$. By Theorem 3.7, the spectrum of a valuative Bézout domain is the union of the (linearly ordered) chain of nonmaximal primes with the set of (at most three) maximal ideals. As R is n-dimensional, if it is valuative, then $|\operatorname{Spec}(R)| = n + |\operatorname{Max}(R)| \leq n + 3$, with at least two maximal ideals of height n when $|\operatorname{Max}(R)| = 3$.

As for the reverse implication in (b), if (i) holds, then R has at most two maximal ideals, the nonmaximal primes of R are linearly ordered by inclusion, and at least one of the maximal ideals contains each of the nonmaximal primes. Hence R is valuative by Theorem 3.7.

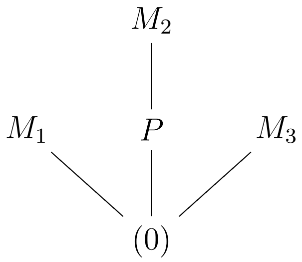
Next, suppose $|\operatorname{Spec}(R)| = n+3$ with $|\operatorname{Max}(R)| = 3$. Then we see easily that the nonmaximal primes of R are linearly ordered by inclusion and therefore each maximal ideal of height n contains each nonmaximal prime. If there are at least two maximal ideals of height n, then R is valuative by Theorem 3.7. □

Consider an n-dimensional Bézout domain R such that $|\operatorname{Spec}(R)| = n+3$. If either R has only two maximal ideals or R has three maximal ideals but only one of them has height n, then more than one maximal ideal of R fails to contain each nonmaximal prime (and so, by Theorem 3.7, R is not valuative). These two cases can be illustrated by the following pictures (with $n = 2$).

(5)



(6)



In both of the above cases, $|\operatorname{Spec}(R)| = n + 3 = 5$. In picture (5), another argument that R is not valuative is that R fails to satisfy the equation $|\operatorname{Spec}(R)| = n + |\operatorname{Max}(R)|$; but note that this equation is satisfied in picture (6). As above, the construction of Bézout domains with spectra as in pictures (5) and (6) is classical.

4. Pointwise minimal extensions

For a pair of rings $R \subsetneq T$, we say that T is a *pointwise minimal extension* of R if $R \subsetneq R[t]$ is a minimal extension for each $t \in T \backslash R$. If, in addition, T is integral over R, then $R \subsetneq R[t]$ is a minimal integral extension for each $t \in T \backslash R$ and we then say that $R \subsetneq T$ is a *pointwise minimal integral extension.*

The notion of pointwise minimal (integral) extension plays a central role in our study, as we have the following necessary condition.

Proposition 4.1. *Let R be a valuative domain. Then either $R = R'$ or $R \subsetneq R'$ is a pointwise minimal* (*integral*) *extension of R.*

Proof. Let $u \in R' \setminus R$. Then $u \in R[u^{-1}]$. As R is valuative and $R \subsetneq R[u]$, this ring extension must then be minimal. □

In this section, we record a few facts about pointwise minimal extensions, with a special focus on pointwise minimal integral extensions.

Subextensions and localization. Our first result about subextensions is trivial.

Proposition 4.2. *Let $R \subsetneq R_1 \subsetneq T$ be three rings such that $R \subsetneq T$ is a pointwise minimal extension* (*resp., a pointwise minimal integral extension*)*. Then $R \subsetneq R_1$ is a pointwise minimal extension* (*resp., a pointwise minimal integral extension*)*.*

Yet, if $R \subsetneq T$ is a pointwise minimal extension and $R \subsetneq R_1 \subsetneq T$, then Example 5.13 shows that $R_1 \subsetneq T$ need not be a pointwise minimal extension. Also, even if both $R \subsetneq R_1$ and $R_1 \subsetneq T$ are pointwise minimal extensions, Example 5.14 shows that $R \subsetneq T$ need not be a pointwise minimal extension. Along these lines, the next result will be useful.

Proposition 4.3. *Let $R \subsetneq T$ be a pointwise minimal extension and I be an ideal of T. Then either $R + I = T$ or $R + I \subsetneq T$ is a pointwise minimal extension.*

Proof. Suppose that $R + I \neq T$. Choose $u \in T \backslash (R + I)$ and $b \in (R + I)[u]$. We will show that either $b \in R + I$ or $u \in (R + I)[b]$. Write $b = f(u)$, with f a polynomial with coefficients in $R + I$. Then $f = g + h$, with g a polynomial with coefficients in R, and h a polynomial with coefficients in I. Thus $b = s + j$, with $s := g(u) \in R[u]$ and $j := h(u) \in I$. By hypothesis, $R \subsetneq R[u]$ is a minimal extension. Hence, either $s \in R$ or $R[s] = R[u]$. In the first case, $b = s + j \in R + I$. In the second case, $u = p(s)$, for some polynomial p with coefficients in R. As $s = b - j$ and $j \in I$, we then have $u = p(b - j) = p(b) + j_1$, with $j_1 \in I$. Thus $u \in (R + I)[b]$. □

We already recalled that, for a minimal extension $R \subsetneq T$ and a multiplicative subset S of R, either $R_S = T_S$ or $R_S \subsetneq T_S$ is a minimal extension. It follows that a similar property obviously holds for pointwise minimal extensions.

Lemma 4.4. *Let $R \subsetneq T$ be a pointwise minimal extension. Then, for each multiplicative subset S of R, either $R_S = T_S$ or $R_S \subsetneq T_S$ is a pointwise minimal extension.*

Crucial maximal ideal. A key property of minimal extensions, already pointed out by Ferrand and Olivier [11, Théorème 2.2 (i)] is that if $R \subsetneq T$ is a minimal ring extension, then there is a (necessarily unique) maximal ideal M of R, called the *crucial* ideal of the extension, such that $R_M \subsetneq T_M$ is a minimal ring extension and $R_N = T_N$ canonically for each prime (equivalently, maximal) ideal N of R that is distinct from M. An easy consequence of globalization is that, conversely, the presence of such a maximal ideal M ensures that a ring extension $R \subseteq T$ is minimal [10, Proposition 2.1]. A similar characterization holds for pointwise minimal extensions. For the sake of simplicity, we restrict to the case where R is a domain and T is an overring of R (as this is the only case that we will need to study valuative domains).

Theorem 4.5. *Let R be a domain and T be a proper overring of R. Then $R \subsetneq T$ is a pointwise minimal extension if and only if there is a maximal ideal M of R such that $R_M \subsetneq T_M$ is a pointwise minimal extension and $R_N = T_N$ canonically for each prime* (*equivalently, maximal*) *ideal N of R such that $N \neq M$.*

Proof. The "if" assertion follows immediately from the above-mentioned observation in [10]. We next prove the contrapositive of the "only if" assertion. Suppose that M and N are distinct maximal ideals of R such that $R_M \neq T_M$ and $R_N \neq T_N$. We claim there is an element $t \in T$ such that $t \notin R_M \cup R_N$. By hypothesis, there exist $a, b \in T$ such that $a \notin R_M$ and $b \notin R_N$. We may assume that $a \in R_N$ and $b \in R_M$ (for, if not, the claim is proved). Then $t := a + b$ satisfies our claim. It follows that $\sqrt{(R :_R t)} \subseteq M \cap N$ and, hence, that $R \subsetneq R[t]$ is not minimal. □

We call the maximal ideal M in Theorem 4.5 the *crucial maximal ideal* of the extension $R \subsetneq T$.

Pointwise minimal integral extensions. In the case of a pointwise minimal integral extension, we next show that the crucial maximal ideal is an ideal of T.

Theorem 4.6. *Let R be a domain and T an overring of R such that $R \subsetneq T$ is a pointwise minimal integral extension. Then the crucial maximal ideal M of the extension is the conductor $(R : T)$. In particular, $M = (R : T)$ is an ideal of T.*

Proof. Assume $R \subsetneq T$ is a pointwise minimal integral extension. For each $t \in T \backslash R$, the crucial maximal ideal M is the only maximal ideal that can contain $(R :_R t)$, since $t \in R_N$ for all other maximal ideals N. As $R \subsetneq R[t]$ is a minimal integral extension, $(R : R[t])$ is a maximal ideal of R [11, Théorème 2.2]. Since $(R : R[t]) \subseteq$

$(R :_R t) \subseteq M$, we must have $(R : R[t]) = M$. It follows that $M = (R : T)$ is an ideal of T. $\square$

In the above situation, R is a pullback, namely, $R = R/M \times_{T/M} T$:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R/M \arrow[d] \\
T \arrow[r] & T/M
\end{tikzcd}$$

The next (easy) result deals with more general pullbacks.

Proposition 4.7. *Let $R \subsetneq T$ be a pair of rings sharing an ideal I. Then $R \subsetneq T$ is a pointwise minimal extension (resp., a pointwise minimal integral extension) if and only if $R/I \subsetneq T/I$ is a pointwise minimal extension (resp., a pointwise minimal integral extension).*

Proof. The rings contained between R and T are in one-to-one correspondence with the rings contained between R/I and T/I; rings of the form $R[t]$ (with $t \in T$) correspond to rings of the form $(R/I)[\bar{t}]$, with $\bar{t} := t + I$. Moreover, T is an integral extension of R if and only if T/I is an integral extension of R/I. $\square$

Let R be a valuative domain that is not integrally closed. We have seen that $R \subsetneq R'$ is a pointwise minimal integral extension (Proposition 4.1). Hence $R_P = R'_P$ for each prime ideal P except for the crucial maximal ideal M. As R' is a Bézout domain (Corollary 3.5), we derive the following.

Corollary 4.8. *Let R be a valuative domain. Then $R_P = R'_P$ is a valuation domain for each prime ideal P of R except for at most one maximal ideal M of R.*

Extensions of a quasilocal ring. If (R, M) is a quasilocal domain and T an overring such that $R \subsetneq T$ is a pointwise minimal extension, then M is obviously the crucial maximal ideal of the extension. We present the next result in this simple situation and then derive a similar result in the global case.

Theorem 4.9. *Let (R, M) be a quasilocal domain, T an overring of R such that $R \subsetneq T$ is a pointwise minimal integral extension, and J the Jacobson radical of T. Then $s^2 \in M$ for each $s \in J$. Moreover, if J is a principal ideal of T or if the field R/M is not of characteristic 2, then $J^2 \subseteq M$.*

Proof. Let $s \in J$. The ring extension $R \subseteq R[s]$ is either minimal or trivial (if $s \in R$), and so $R[s^2]$ is either R or $R[s]$. If $R[s^2] = R$, then s^2 is a nonunit of R, i.e., $s^2 \in M$. If $R[s^2] = R[s]$, we can write $s = r_0 + r_2 s^2 + \cdots + r_{2n} s^{2n}$, for some elements $r_i \in R$. Thus $r_0 = s(1 + ws)$, with $w \in J$. As J is the Jacobson radical of T, $1 + ws$ is a unit, and so s and r_0 are associates in T. Therefore, $r_0 \in R \cap J = M$, and hence $s \in M$ (since M is an ideal of T). If moreover J is principal, it follows immediately that $J^2 \subseteq M$. Finally, suppose that the field R/M is not of characteristic 2. Let $s, t \in J$. Then s^2, t^2, and $(s + t)^2$ belong to M, and so $2st \in M$. We can conclude that $st \in M$ (since 2 is a unit of T/M). □

In the global case, as $R_N = T_N$ for each maximal ideal N distinct from the crucial maximal ideal M, globalization leads to similar conclusions for the intersection of the maximal ideals of T that contain M (*a fortiori*, for every smaller ideal, in particular for the Jacobson radical of T).

Corollary 4.10. *Let R be a domain, T an overring of R such that $R \subsetneq T$ is a pointwise minimal integral extension, M the crucial maximal ideal of the extension, and J the intersection of the maximal ideals of T that contain M. Then $s^2 \in M$ for each $s \in J$.*

Examples 4.12 below shows that, even in the local case, it does not follow that $J^2 = M$ (it may be that J^2 is not contained in M or that J^2 is strictly contained in M). But first we give a partial converse of Theorem 4.9.

Proposition 4.11. *Let (R, M) be a quasilocal domain and T an overring of R such that M is an ideal of T. Suppose J is an ideal of T such that $s^2 \in M$ for each $s \in J$. If $s \in J$, then $R[s] = R + Rs$. Furthermore, either $R = R + J$ or $R \subsetneq R + J$ is a pointwise minimal integral extension.*

Proof. Let $s \in J$. For each $k \geq 2$, we have $s^k \in M$. Thus $R[s] = R + Rs$. Assuming $R \neq R + J$, it remains to show that, if $b \in (R + J) \setminus R$, then $R \subsetneq R[b]$ is a minimal extension (as $R + J$ is obviously integral over R). Writing $b = r + s$, $r \in R, s \in J$, then $R[b] = R[s]$, and we may replace b with s. Let $t \in R[s]$. Then $t = a + ws$, with $a, w \in R$. If $w \in M$, we have $t \in R$, and so $R = R[t]$. If $w \notin M$, then w is a unit of R and $s = w^{-1}(t - a)$, whence $R[s]$ is contained in (and equal to) $R[t]$. □

Examples 4.12. Let F be a field and let $T := F[X,Y]_{(X,Y)}$ be the localization of the polynomial ring $F[X,Y]$ at the maximal ideal (X,Y). Then T is a local domain, with maximal ideal $J := (X,Y)T$, and $T = F + J$. In both examples 1) and 2) below, choosing a proper ideal M of T whose radical is J, we set $R := F + M$. Then (R,M) is a quasilocal domain with maximal ideal M (cf. [12, Theorem 1.4]), R is contained in T, and $T = R + J$.

1) Take $M = (X^2, Y^2)$. If F is a field of characteristic 2, then $s^2 \in M$ for each $s \in J$. It follows from Proposition 4.11 that $R \subsetneq T$ is a pointwise minimal integral extension. Yet J^2 is not in contained in M, as $XY \notin M$.

2) Take $M = (X, XY, Y^2)$. Without any hypothesis on F, we then have $s^2 \in M$ for each $s \in J$. It follows again that $R \subsetneq T$ is a pointwise minimal integral extension. Here, $J^2 \subsetneq M$ as $X \notin J^2$.

The next result is specific to the quasilocal case.

Proposition 4.13. *Let (R,M) be a quasilocal domain that is not integrally closed. If R' is a pointwise minimal extension of R, then each minimal overring of R is contained in R'.*

Proof. Suppose, on the contrary, that $R \subsetneq R[z]$ is a minimal overring with $z \notin R'$. Then $R \subsetneq R[z]$ is a closed minimal extension. As R is quasilocal, $z^{-1} \in M$ (by the proof that (1) $\Rightarrow$ (2) in [9, Theorem 3.1]). Since M is an ideal of R' by Theorem 4.6, we thus have $z^{-1}t \in R$ for each $t \in R'$, and so $R \subsetneq R' \subsetneq Rz \subseteq R[z]$. This contradicts the minimality of $R \subsetneq R[z]$. □

In view of [20, page 1738], the proof of Proposition 4.13 establishes the following more general result. If R is a quasilocal domain that is not integrally closed and R' is a pointwise minimal extension of R, then each minimal domain extension of R is, up to R-algebra isomorphism, contained in R'.

By combining Proposition 4.13 with Proposition 4.1 it follows that if (R,M) is a quasilocal non-integrally closed valuative domain, then R' contains every minimal overring of R. In case R is not quasilocal, we shall however give an example of a non-integrally closed valuative domain with a closed minimal overring (Example 6.7).

Pointwise minimal field extensions. We close this section with a characterization of the pointwise minimal field extensions. Note that, for a pair of fields, if $F \subsetneq L$ is a pointwise minimal extension, then L is clearly algebraic over F and, hence, every intermediate ring is a field.

Lemma 4.14. *Let $F \subsetneq L$ be fields of prime characteristic p and let $r, s \in L \backslash F$ be algebraic such that r is separable over F and $s \in F^{1/p}$. Then $F[r, s] = F[r + s]$ and $F \subsetneq F[r] \subsetneq F[r + s]$.*

Proof. Since r is separable over F, the field extension $F[r + s] \subseteq F[r + s][r] = F[r, s] = F[r + s][s]$ is separable. As s is both separable and purely inseparable over $F[r + s]$, we have $s \in F[r + s]$, and thus both r and s are in $F[r + s]$. Hence $F[r + s] = F[r, s]$. Finally, $F[r] \neq F[r + s]$ since the separable field extension $F \subsetneq F[r]$ cannot contain (the purely inseparable element) s. □

Lemma 4.15. *Let $F \subsetneq L$ be a pair of fields of prime characteristic p, and let $s \in L$ be algebraic over F such that $F[s^p] = F[s]$. Then s is separable over F.*

Proof. By hypothesis, $s = \alpha_0 + \alpha_1 s^p + \ldots + \alpha_n s^{p^n}$, with coefficients $\alpha_i \in F$. Thus s is a root of the polynomial $g(X) := \alpha_0 - X + \alpha_1 X^p + \ldots + \alpha_n X^{p^n}$. The derivative of $g(X)$ is $g'(X) = -1$. As s is not a root of $g'(X)$, s must be a simple root of $g(X)$. *A fortiori*, s is a simple root of its minimal polynomial. □

Proposition 4.16. *Let $F \subsetneq L$ be a pair of fields. Then $F \subsetneq L$ is a pointwise minimal extension if and only if either $F \subsetneq L$ is a (necessarily algebraic) minimal field extension or F has prime characteristic p such that $L \subseteq F^{1/p}$.*

Proof. A minimal (necessarily algebraic) field extension is obviously a pointwise minimal extension. In the prime characteristic case with $L \subseteq F^{1/p}$, then each $t \in L \setminus F$ satisfies $[F[t] : F] = p$, and so $F \subsetneq F[t]$ is minimal.

Conversely, assume that $F \subsetneq L$ is a pointwise minimal extension. Then L is algebraic over F. We consider two cases, depending on whether or not $F \subsetneq L$ is a separable extension.

— If $F \subsetneq L$ is a separable extension, the Primitive Element Theorem yields that each finite subextension of $F \subseteq L$ can be generated as an F-algebra by one element. The "pointwise minimal extension" hypothesis then shows that L must be a minimal field extension of F.

— If $F \subsetneq L$ is not a separable extension, then F has prime characteristic p. Choose an element $s \in L$ that is not separable over F. By Lemma 4.15, $F[s^p] \subsetneq F[s]$. If $s \notin F^{1/p}$, then $F \subsetneq F[s^p] \subsetneq F[s]$, contradicting the "pointwise minimal extension" hypothesis. Hence $s \in F^{1/p}$. It suffices to prove that $F \subsetneq L$ is a purely inseparable extension (for then $L \subseteq F^{1/p}$). Suppose not. Then there exists $r \in L \backslash F$ such that r is separable over F. Then Lemma 4.14 yields the containments $F \subsetneq F[r] \subsetneq F[r, s] = F[r+s]$ (with $s \notin F[r]$ since $F \subsetneq F[r]$ is separable while $F[r+s]$ contains s which is not separable over F). This contradicts the "pointwise minimal extension" hypothesis. □

5. The quasilocal case

We now consider the case where the valuative domain (R, M) is quasilocal and not necessarily integrally closed. By Proposition 3.1, R' is a valuation domain; and if $R \neq R'$, then $R \subsetneq R'$ is a pointwise minimal extension by Proposition 4.1. In fact, these two conditions are jointly sufficient and thus provide a first characterization.

Proposition 5.1. *Let (R, M) be a quasilocal domain. Then R is valuative if and only if R' is a valuation domain such that either $R = R'$ or $R \subsetneq R'$ is a pointwise minimal extension.*

Proof. By the above comments, it remains to show that if R' is a valuation domain and $R \subsetneq R'$ is a pointwise minimal extension, then R is valuative. Let $0 \neq z \in K$ such that neither z nor z^{-1} is in R. Then at least one of z, z^{-1} is in $R' \setminus R$. As $R \subsetneq R[u]$ is a minimal extension for each $u \in R' \backslash R$, it follows that R is valuative. □

In relation with Corollary 3.3, note that in Proposition 5.1, we can replace the condition that R' is a valuation domain with the condition that R is an i-domain.

If R' is a valuation domain and $R \subsetneq R'$ is a pointwise minimal (integral) extension, we know by Theorem 4.6 that the crucial maximal ideal $M = (R : R')$ is an ideal of R' and hence that R is the pullback $R = R/M \times_{R'/M} R'$:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R/M \arrow[d] \\
R' \arrow[r] & R'/M
\end{tikzcd}$$

We will split the characterization of the quasilocal valuative domains into several cases. First, we suppose that the above ideal M is the maximal ideal of R', that is, that R is a pseudo-valuation domain. Next, assuming that the maximal ideal N of R' strictly contains M, we first consider the subcase where $R' = R + N$, and then give a general characterization of the quasilocal valuative domains that are not pseudo-valuation domains.

Pseudo-valuation domains. We let R be a pseudo-valuation domain, that is, a quasilocal domain (R, M) sharing its maximal ideal M with a valuation overring V. Then $R \subseteq R' \subseteq V$. Furthermore, R' is quasilocal with maximal ideal M, that is, R' is a pseudo-valuation domain with associated valuation domain V. Consider the fields $k := R/M$, $L := R'/M$ and $F := V/M$. Then $k \subseteq L \subseteq F$ and L is the algebraic closure of k in F (cf. [12, Corollary 1.5 (5)]). Also, R' is a valuation domain if and only if $R' = V$, that is, if and only if $L = F$. We next give several equivalent conditions for R to be valuative.

Theorem 5.2. *Let (R, M) be a pseudo-valuation domain, with canonically associated valuation overring (V, M). Consider the fields $k := R/M$, $L := R'/M$ and $F := V/M$. Then the following conditions are equivalent:*

(1) *R is valuative;*

(2) *$R' = V$ and either $R = R'$ $(= V)$ or $R \subsetneq R'$ $(= V)$ is a pointwise minimal extension;*

(3) *$L = F$ and either $k = L$ $(= F)$ or $k \subsetneq L$ $(= F)$ is a pointwise minimal field extension;*

(4) *either $k = F$ or $k \subsetneq F$ is a pointwise minimal field extension;*

(5) *either $k = F$, or the extension $k \subsetneq F$ satisfies one of the following two conditions:*
   (i) *$k \subsetneq F$ is a minimal field extension,*
   (ii) *k has prime characteristic p such that $k \subsetneq F \subseteq k^{1/p}$.*

Proof. (1) ⇔ (2): Combine Proposition 5.1 with the fact that R' is a valuation domain if and only if $R' = V$.

(2) ⇔ (3): Clearly $R' = V$ (resp., $R = R'$) if and only if $L = F$ (resp., $k = L$); and it follows by Proposition 4.7 that $R \subsetneq V$ is a pointwise minimal extension if and only if $k \subsetneq F$ is a pointwise minimal extension.

(3) $\Rightarrow$ (4) is obvious and (4) $\Leftrightarrow$ (5) is Proposition 4.16.

((5) and (4)) $\Leftrightarrow$ (3): It is clear that each condition in (5) implies that F is algebraic over k, that is, $L = F$. Under this condition, (4) and (3) become obviously equivalent. $\square$

For a given pseudo-valuation domain R, it is interesting to note (using the above notation) that one cannot drop the requirement that $L = F$ in condition (3) of Theorem 5.2. Indeed, R can be integrally closed (equivalently, $k = L$) or $R \subsetneqq R'$ can be a pointwise minimal extension (equivalently, $k \subsetneqq L$ be a pointwise minimal extension) while $R' \neq V$ (that is, while R' is not a valuation domain). In this situation, R fails to be valuative. For an example of this, consider the pseudo-valuation domain $R := \mathbb{R} + X\mathbb{C}(Y)[[X]]$. Then $R' = \mathbb{C} + X\mathbb{C}(Y)[[X]]$ is a minimal extension of R (since $R'/M \cong \mathbb{C}$ is a minimal extension of $\mathbb{R} \cong R/M$) while $V = \mathbb{C}(Y)[[X]] \supsetneqq R'$. This example also shows that a pseudo-valuation domain that has a minimal overring need not be valuative.

The following corollary splices together portions of Proposition 5.1 and Theorem 5.2 to specifically characterize the valuative pseudo-valuation domains that are not integrally closed.

Corollary 5.3. *Let (R, M) be a pseudo-valuation domain that is not integrally closed. Then R is valuative if and only if R' is a valuation domain (with maximal ideal M) such that either $R/M \subsetneqq R'/M$ is a minimal algebraic extension of fields or R/M has prime characteristic p such that $R'/M \subseteq (R/M)^{1/p}$.*

We can now describe the overrings of valuative pseudo-valuation domains and show they are all valuative.

Corollary 5.4. *Let (R, M) be a valuative pseudo-valuation domain, with canonically associated valuation overring (R', M), and let S be an overring of R. Then S is a valuative pseudo-valuation domain. Moreover, S is of exactly one of the following two types:*

1. *$R' \subsetneqq S$: then S is a valuation domain and $S = R_P$ for some nonmaximal prime ideal P of R;*
2. *$S \subseteq R'$: then S is a pseudo-valuation domain with canonically associated valuation overring (R', M).*

Proof. Any overring of a pseudo-valuation domain D is comparable with the canonically associated valuation overring of D (cf. the proof of [3, Theorem 3.1] or [8, Theorem 2.1 (a)]). Thus, either $R' \subsetneq S$ or $S \subseteq R'$. If $R' \subsetneq S$, then S is a valuation domain whose maximal ideal lies over a nonmaximal prime ideal P of R. Since R_P is a valuation domain by Corollary 4.8 and S is a quasilocal domain that dominates R_P, it follows that $S = R_P$ is a valuative pseudo-valuation domain. If $S = R'$, then S is a valuation domain, hence a valuative pseudo-valuation domain.

It remains only to consider the case where $S \subsetneq R'$. In this case, $S' = R'$ and as R and R' share the maximal ideal M, S is also quasilocal with maximal ideal M. Therefore, S is a pseudo-valuation domain with canonically associated valuation overring (R', M). As R is valuative, it follows from the characterization of valuative pseudo-valuation domains in Theorem 5.2 that either $R/M \subsetneq R'/M$ is a minimal (algebraic) field extension or R/M has prime characteristic p such that $R'/M \subseteq (R/M)^{1/p}$. In view of the inclusions $R/M \subseteq S/M \subsetneq R'/M$, we see that either $S/M \subsetneq R'/M$ is a minimal (algebraic) field extension or S/M has characteristic p such that $R'/M \subseteq (S/M)^{1/p}$. Another application of Theorem 5.2 shows that S is valuative. □

Quasilocal non-pseudo-valuation domains, special case. If (R, M) is a valuative quasilocal domain, we know that R' is a valuation domain. We now consider the case where the maximal ideal N of R' strictly contains M. We first show there is a necessary condition on the valuation domain R' in this case (something that had no parallel in the case of a pseudo-valuation domain).

Lemma 5.5. *Let (R, M) be a quasilocal valuative domain that is neither integrally closed nor a pseudo-valuation domain. Then R' is a valuation domain, the maximal ideal N of R' is principal, and $N^2 = M$.*

Proof. By Propositions 3.1 and 4.1, R' is a valuation domain and $R \subsetneq R'$ is a pointwise minimal extension of R. Since R is not a pseudo-valuation domain, the maximal ideal N of R' properly contains M, and by Theorem 4.9, $s^2 \in M$ for each $s \in N$. Thus $N^2 \neq N$. Since R' is a valuation domain, N must be principal. Then $N^2 \subseteq M$. As the maximal ideal N is principal, no ideal of the quasilocal domain R' can lie properly between N^2 and N. Since M is an ideal of R' (Theorem 4.6), it follows that $M = N^2$. □

The first conclusion in the following lemma is closely related to [9, Lemma 2.11].

Lemma 5.6. *Let (R, M) be a quasilocal domain which is not integrally closed and is such that R' is a valuation domain with principal maximal ideal N and $M = N^2$. Then $R \subsetneq R + N$ is a pointwise minimal extension. Moreover, if $s \in N \backslash M$, then s is a generator of the principal ideal N and $R[s] = R + Rs$.*

Proof. The hypotheses ensure that $N \neq M$. Hence by Proposition 4.11 (with $T := R'$ and $J := N$), it follows that $R \subsetneq R + N$ is a pointwise minimal extension and that $R[s] = R + Rs$ for each $s \in N$. It remains to show that if $s \in N \backslash M$, then $R's = N$. If this fails, then $s \in N^2 = M$, the desired contradiction. $\square$

In the above context, if $R + N = R'$ we can conclude that R is valuative from Proposition 5.1. We next give several characterizations of this equality.

Proposition 5.7. *Let (R, M) be a quasilocal domain which is not integrally closed and is such that R' is a valuation domain with principal maximal ideal N and $M = N^2$. Then the following conditions are equivalent:*

1. *$R' = R + N$;*
2. *the canonical ring homomorphism $R/M \to R'/N$ is an isomorphism;*
3. *$R' = R[s]$ for each $s \in N \backslash M$;*
4. *$R' = R[s]$ for some $s \in N$;*
5. *R' is a minimal extension of R.*

Under these conditions, R is valuative and R' is the only minimal overring of R.

Proof. Obviously (1) $\Leftrightarrow$ (2), using the natural identification $R/M = (R + N)/N$.
(1) $\Rightarrow$ (3): If $R' = R + N$ and $s \in N \backslash M$, we infer the following from Lemma 5.6:

$$R' = R + N = R + R's = R + (R + N)s = R + Rs = R[s].$$

(3) $\Rightarrow$ (4): This is clear because the hypotheses ensure that $N \neq M$.
(4) $\Rightarrow$ (1): If $R' = R[s]$ for some s in N, then $R' = R + N$ since N is an ideal of R'.
(3) $\Rightarrow$ (5): Assuming (3), we show that $R' = R[b]$ for each $b \in R' \backslash R$ and, hence, that $R \subsetneq R'$ is a minimal extension. As we have now proved that (3) $\Leftrightarrow$ (1), we can write $b = r + n$, for some $r \in R$ and $n \in N$. Clearly, $R[b] = R[n]$. As $b \notin R$, we have $n \in N \backslash M$. Thus, by (3), $R' = R[n] = R[b]$.
(5) $\Rightarrow$ (3): Clear.

Under conditions (1)–(5), Proposition 5.1 yields that R is valuative and Proposition 4.13 yields that R' is the only minimal overring of R. $\square$

Just as we did for valuative pseudo-valuation domains, we next describe the overrings of a quasilocal valuative domain R of the type described in Proposition 5.7 and show that they are all valuative.

Corollary 5.8. *Let* (R, M) *be a quasilocal valuative domain that is neither integrally closed nor a pseudo-valuation domain and is such that* $R' = R + N$ *with* N *the* (*principal*) *maximal ideal of* R'*. Then each proper overring of* R *is a valuation domain and, except for* R'*, is of the form* R_P *for some nonmaximal prime* P *of* R*. In particular, each overring of* R *is valuative.*

Proof. By Lemma 5.5, R' is a valuation domain with principal maximal ideal N where $M = N^2$. This occurs no matter whether $R' = R + N$ or not, but in this special case R' is the only minimal overring of R by Proposition 5.7. Let S be a proper overring of R other than R'. Then $S' \supseteq R'$. As there are no rings properly between R and R', S' properly contains R'. It follows that S' is a valuation domain whose maximal ideal lies over some nonmaximal prime P of R. Thus, S is quasilocal and its maximal ideal lies over P. Note that Corollary 4.8 gives that R_P is a valuation domain. Because this domain is dominated by the quasilocal domain S, we have $S = R_P (= S')$. $\square$

Quasilocal non-pseudo-valuation domains, general result. We now give a (complete) characterization of the quasilocal valuative domains that are not pseudo-valuation domains (and thus not integrally closed).

Lemma 5.9. *Let* (R, M) *be a quasilocal valuative domain that is not a pseudo-valuation domain, and let* N *be the maximal ideal of* R'*. Then* $R + N$ *is a valuative pseudo-valuation domain.*

Proof. $R + N$ is quasilocal (cf. [12, Theorem 1.4]) with maximal ideal N, hence a pseudo-valuation domain with canonically associated valuation overring R'. Also, by Proposition 4.1, $R \subsetneq R'$ is a pointwise minimal extension. Hence, by Proposition 4.3, either $R + N = R'$ or $R + N \subsetneq R'$ is a pointwise minimal (integral) extension. As $(R + N)' = R'$, it now follows via Proposition 5.1 that $R + N$ is valuative. $\square$

Theorem 5.10. *Let (R, M) be a quasilocal domain that is not a pseudo-valuation domain. Then the following conditions are equivalent:*

1. *R is a valuative domain;*
2. *R' is a valuation domain with principal maximal ideal N, $M = N^2$ and if $R + N \neq R'$, then R/M has prime characteristic p and $u^p \in R$ for each $u \in R'$.*

Proof. (1) $\Rightarrow$ (2): Assume (1). By Lemma 5.5, R' is a valuation domain with principal maximal ideal N and $M = N^2$. Suppose that $R+N \neq R'$. By Lemma 5.9, $R + N$ is a valuative pseudo-valuation domain. From the characterization given in Theorem 5.2, it follows that either $((R + N)/N \cong)$ $R/M \subsetneq R'/N$ is a (necessarily algebraic) minimal field extension or R/M has prime characteristic p such that $R'/N \subseteq (R/M)^{1/p}$. In fact, we show next that the extension $R/M \subsetneq R'/N$ is purely inseparable, and hence that we are in the case where R/M has prime characteristic p, and that $R'/N \subseteq (R/M)^{1/p}$.

We claim that R/M is separably closed in R'/N. Suppose that this claim fails. Then there exists $c \in R'\backslash(R + N)$ such that $\overline{c} := c + N \in R'/N$ is separable over R/M. If a polynomial $f \in R[X]$ is such that $f(c) \in R + N$, then $f(c) \in R$ by Proposition 4.1, since $R \subseteq R[f(c)] \subsetneq R[c]$. Hence if $f \in R[X]$ is a monic polynomial of smallest degree such that $f(c) \in R$, then the canonical image $\overline{f}$ of f in $(R/M)[X]$ is the minimal polynomial of $\overline{c}$. Since a separable polynomial and its derivative cannot have a common root, we see that $f'(c) \notin N$, and so $f'(c)$ is a unit of R'. Next, choose $s \in N \setminus M$. As M is an ideal of R' (by Proposition 4.1 and Theorem 4.6), we have $f'(c)s \in N \setminus M$. By the formal Taylor Theorem (or a direct calculation), $f(c+s) = f(c) + f'(c)s + m$ for some $m \in s^2R' \subseteq M$. It follows that $f'(c)s \in R[c + s]$. In fact, $R \subsetneq R[f'(c)s] \subseteq R[c + s]$. As $R \subsetneq R'$ is a pointwise minimal extension, $R[f'(c)s] = R[c + s]$, whence $c + s \in R + N$, contradicting that $c \notin R + N$. This proves the above claim. Since $R/M \subseteq R'/N$ is an algebraic field extension that is separably closed, it must be purely inseparable, of some prime characteristic p.

We now prove that $u^p \in R$ for each $u \in R'$. Since R is quasilocal with maximal ideal M and the characteristic of R/M is p, we have $p \in M$. Thus, if $u \in R + N$, we see that $u^p \in R + pN + N^2 = R + M = R$. It remains to show that if $u \in R'\backslash(R+N)$, then $u^p \in R$. Suppose by way of contradiction that $u^p \notin R$. Then

$R \subsetneq R[u^p] \subsetneq R[u]$, since $u^p \in R+N$ while $u \notin R+N$. This contradicts that $R \subsetneq R'$ is a pointwise minimal extension.

(2) $\Rightarrow$ (1): Assume (2). If $R' = R + N$, then R is valuative by Proposition 5.7. Thus, we may suppose that R/M has prime characteristic p and that $u^p \in R$ for each $u \in R'$. To prove that R is valuative, it suffices by Proposition 5.1 to show that $R \subsetneq R[u]$ is a minimal extension for each $u \in R' \backslash R$. This follows immediately from Lemma 5.6 if $u \in R + N$. Thus, without loss of generality, $u \in R' \backslash (R + N)$.

Let us first show that if $n \in N$ is such that $n \in R[u]$, then $n \in M$. Note that $\overline{u} := u + N \in R'/N$ is purely inseparable over R/M, with minimal polynomial $X^p - \overline{u}^p$. As $n \in R[u]$, one can write $n = g(u)$ with $g \in R[X]$. Since $u^p \in R$, we may assume (via the division algorithm in $R[X]$, upon dividing g by $X^p - u^p$ and replacing g with the remainder) that $\deg(g) < p$. In R'/N, we have $\overline{g}(\overline{u}) = \overline{n} = \overline{0}$, with $\overline{g} \in (R/M)[X]$. It follows that $\overline{g}$ is the zero polynomial, whence all the coefficients of g are in M. Hence, $n \in MR' = M$.

It suffices to prove that if $w \in R[u]$, then either $w \in R$ (and hence $R[w] = R$) or $u \in R[w]$ (and hence $R[w] = R[u]$). As $u^p \in R$ for each $u \in R'$, we have $R'/N \subseteq (R/M)^{1/p}$. As $R + N$ is a pseudo-valuation domain, Theorem 5.2 yields that $R + N$ is valuative. Hence, by Proposition 4.1, $R + N \subsetneq R'$ is a pointwise minimal extension. As $w \in R[u] \subseteq (R+N)[u]$, either $w \in R+N$ or $u \in (R+N)[w]$.
— If $w \in R + N$, write $w = r + n$, with $r \in R$ and $n \in N$. As $w \in R[u]$, we then have $n \in R[u]$. We showed above that this implies $n \in M$. Thus, $w = r + n \in R$ in this case.
— In the remaining case, $u \in (R+N)[w]$. Write $u = h(w) + n$ for some polynomial $h \in R[X]$ and some element $n \in N$. Then $n = u - h(g(u))$, recalling that $w = g(u)$ with $g \in R[X]$. Hence $n \in R[u]$. Once again, this implies $n \in M$. We then have $u = h(w) + n \in R[w]$ in this case, to complete the proof. □

Examples 5.11. 1) Examples of valuative domains, as described in Theorem 5.10, can be given for the case where $R' = R + N$ by pullbacks of the following type:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R/M \cong L \arrow[d] \\
V \arrow[r] & V/M
\end{tikzcd}$$

where V is a valuation domain with principal maximal ideal $N \neq (0)$, $M = N^2$ and $V/N = L$. Then $R' = V = R + N$.

For instance, take $V := L[[X]]$ and $R := L + X^2L[[X]]$. In this example, $R' = V$ is a discrete valuation domain and $R'/M \cong L[X]/(X^2)$.

There is no restriction on the field L in this case.

2) Examples illustrating Theorem 5.10 for the case where $R + N \subsetneq R'$ can be given by pullbacks of the following type:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R/M \cong F \arrow[d] \\
V \arrow[r] & V/M
\end{tikzcd}$$

where V is a valuation domain with principal maximal ideal $N \neq (0)$, $M = N^2$, $V/N = L$ is a field of prime characteristic p, and F is a proper subfield of L with $L \subseteq F^{1/p}$. Then $R' = V$, but now $R' \supsetneq R + N$.

For instance, take (as before) $V := L[[X]]$, but now let $R := F + X^2L[[X]]$, with $L = k(Y)$, where k is a field of prime characteristic p, Y an indeterminate, and $F = k(Y^p)$. Then $R' = V \supsetneq R + N$ and $R/M \subsetneq R'/N \subseteq (R/M)^{1/p}$.

As $F \subsetneq L$ is a purely inseparable extension, note that F (and, *a fortiori*, L) must be infinite in this case.

Overrings. Let R be a quasilocal valuative domain that is not a pseudo-valuation domain, with N the unique maximal ideal of R'. When $R' = R+N$, we have seen in Proposition 5.7 that R' is the only minimal overring of R. In case $R + N \subsetneq R'$, we show next that there are infinitely many distinct minimal overrings of R. Moreover, infinitely many of those contained in $R + N$ are not valuative. Thus the valuative property need not be inherited by integral overrings.

Proposition 5.12. *Let (R, M) be a quasilocal valuative domain that is not a pseudo-valuation domain and let N be the maximal ideal of R'. If $R + N \subsetneq R'$, then the minimal overrings of R are of two types:*

(1) *Those contained in $R + N$, necessarily of the type $R[s]$ with $s \in N \setminus M$. There are infinitely many distinct such (minimal) overrings of R, and none of them is valuative.*

(2) *Those not contained in* $R+N$, *of the type* $R[u]$ *with* $u \in R' \backslash (R+N)$ *a unit of* R'. *All such* (*minimal*) *overrings of* R *are valuative.*

Proof. By Propositions 4.13 and 4.1, each minimal overring of R is contained in R' and hence is of the form $R[b]$ for some $b \in R' \backslash R$; and each such $R[b]$ is a minimal overring of R. If $b = r + s \in R + N$ with $r \in R$ and $s \in N$, then $R[b] = R[s]$; and if $b \in R' \setminus (R + N)$, then b is a unit of R', as $b \notin N$. Thus the minimal extensions of R that are contained in $R + N$ are of the form $R[s]$ with $s \in N \backslash M$, while the minimal extensions of R that are not contained in $R+N$ are of the form $R[u]$ with $u \in R' \setminus (R + N)$, necessarily a unit of R'.

Let us show there are infinitely many distinct minimal extensions of the first type. Choose u in $R' \backslash (R + N)$ and s in $N \backslash M$. If $w \in M$, then $u + w \notin R + N$, whence $u+w$ is a unit of R'; as M is an ideal of R', it follows that $(u+w)s \in N \backslash M$, and so $R \subsetneq R[(u+w)s]$. We claim that if $w_1, w_2 \in R$ are in distinct cosets modulo M, then $R[(u+w_1)s] \neq R[(u+w_2)s]$. As R/M is infinite (since R'/N is a purely inseparable extension of R/M by the proof of Theorem 5.10), it follows from the claim that there are infinitely many distinct minimal extensions of R that are contained in $R + N$.

Suppose that $R[(u + w_1)s] = R[(u + w_2)s]$ for some $w_1, w_2 \in R$. In particular, $(u+w_1)s \in R[(u+w_2)s]$. As $N^2 = M$, we can write $(u+w_1)s = a+b(u+w_2)s$, with $a, b \in R$. Then $a \in N \cap R = M = N^2$. Hence $a = ns$ for some $n \in N$, and so we have $u + w_1 = n + b(u + w_2)$. Rewriting yields $(1 - b)u = n + bw_2 - w_1$. If $(1 - b)$ were a unit of R, we would have $u \in R+N$, contrary to our choice of u. Thus $(1-b) \in M$; that is, $b = 1-m$, with $m \in M$. Therefore, $w_2 - w_1 = mu - n + mw_2 \in N \cap R = M$. Thus w_1, w_2 are congruent modulo M, proving the above claim.

It remains to show that no extensions of the first type are valuative while all the extensions of the second type are valuative. We will use the following easy consequence of integrality: if T is any ring such that $R \subseteq T \subseteq R'$, then T is quasilocal with maximal ideal $N \cap T$. It follows from Proposition 5.7 that if $s \in N \backslash M$, then $R \subsetneq R[s] \subsetneq R + N$. The ring $R[s]$ is quasilocal, with maximal ideal $M_s := N \cap R[s]$. Moreover $M \subsetneq M_s \subsetneq N$. (Indeed, $M \neq M_s$ since $s \in M_s \backslash M$, and $M_s \neq N$ since $R[s] \neq R + N$.) Hence, M_s is not an ideal of R' (for it follows easily from the fact that N is a principal ideal of the quasilocal domain R' that no ideal of R' can be contained strictly between N and $N^2 = M$). By Theorem 4.6 and Proposition 4.1, we can conclude that $R[s]$ is not valuative.

Finally, consider $u \in R' \backslash (R+N)$. We claim that $M = N \cap R[u]$. One inclusion is clear. Conversely, let $m \in R[u] \cap N$. Then $R[m] \subsetneq R[u]$, as $m \in N$ but $u \notin R+N$. Since $R[u]$ is a minimal extension of R, it follows that $m \in R \cap N = M$, thus proving the claim. Hence the ring $R[u]$ is quasilocal with maximal ideal $M = N^2$. (The fact that $R[u]$ is quasilocal also follows because it is an overring of the quasilocal i-domain R.) As the integral closure of $R[u]$ is R' and $b^p \in R$ (*a fortiori*, $b^p \in R[u]$) for each $b \in R'$, it follows from Theorem 5.10 that $R[u]$ is valuative. $\square$

Examples.

Example 5.13. As in Example 5.11.2, let $V := L[[X]]$ and $R := F + X^2L[[X]]$, where L is a field of characteristic p, F a proper subfield of L, and $L \subseteq F^{1/p}$. Then R is a quasilocal valuative domain that is not a pseudo-valuation domain and $R + N \subsetneq R' = V$ (where $N = XL[(X]]$ is the maximal ideal of R'). By Proposition 5.12, the overring $R_1 := R[X] = F + FX + X^2L[[X]]$ (among infinitely many others) is not valuative. For specific "bad" simple extensions, note that for each $y \in L \backslash F$, $R_1 \subsetneq R_1[yX] \subsetneq R_1[y] = F[y] + F[y]X + X^2L[[X]]$. Note this also provides an example where $R \subsetneq T = R'$ is a pointwise minimal (integral) extension but, for some intermediate ring R_1, $R_1 \subsetneq T$ is not a pointwise minimal extension.

The following example shows that in the prime characteristic case, the requirement that $b^p \in R$ for each $b \in R'$ (for R to be valuative) cannot be replaced by simply having $R/M \subsetneq R'/N \subseteq (R/M)^{1/p}$. The problem arises because R/M embeds as $(R+N)/N$ in R'/N. Thus there may be an element $b \in R' \backslash (R+N)$ such that $b^p \in (R+N) \backslash R$. It also provides an example where both $R \subsetneq R_1$ and $R_1 \subsetneq T$ are pointwise minimal extensions while $R \subsetneq T$ is not.

Example 5.14. Let $V := L[[X]]$ with $L := k(Y)$, where k is a field of prime characteristic p and Y is an indeterminate. As in the previous examples, V is a (discrete) valuation domain with principal maximal ideal $N = XL[[X]]$. We set $\beta := Y^p + X$. Then β is a unit of V. Moreover, we claim that for every nonzero polynomial g with coefficients in k, $g(\beta)$ is a unit of V. Indeed, $g(\beta) = g(Y^p + X) = g(Y^p) + Xh(X, Y^p)$, where h is a polynomial in two indeterminates with coefficients in k; and $g(Y^p) \neq 0$ since Y is transcendental over k. This proves the above claim. Thus $k(\beta)$ is contained in V. Set $R := k(\beta) + N^2 = k(\beta) + X^2L[[X]]$. Then $V = R'$ and $M := N^2$ is the maximal ideal of R. The field R'/N is isomorphic to $L = k(Y)$, while $R/M \cong k(\beta)$ (where β is transcendental over k and can be viewed as an indeterminate). As $Y^p \equiv \beta \pmod{N}$, we have $R'/N \subseteq (R/M)^{1/p}$. However $Y^p = \beta - X$ is not in R. Thus, by Theorem 5.10, R is not valuative.

For a specific "bad" simple extension, note that $R \subsetneq R[Y^p] \subsetneq R[Y]$. Indeed, $R[Y^p]$ contains $\beta - Y^p = X$ but does not contain Y.

We can make two further comments:

1) We are in the case where $R \subsetneq R + N \subsetneq R'$. Moreover, both $R \subsetneq R + N$ and $R + N \subsetneq R'$ are pointwise minimal extensions, while $R \subsetneq R'$ is not. For $R \subsetneq R + N$, the assertion follows from Lemma 5.6. On the other hand, $R + N$ is a pseudo-valuation domain and $R'/N \subseteq (R/M)^{1/p}$, and so $R + N$ is valuative by Theorem 5.2. Hence, by Proposition 5.1, $R + N \subsetneq R'$ is a pointwise minimal extension. The same result shows that $R \subsetneq R'$ is not a pointwise minimal extension, since R is not valuative.

2) R is given by the pullback

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R/M \cong F \arrow[d] \\
R' = V \arrow[r] & V/M \cong L[X]/(X^2)
\end{tikzcd}$$

where the embedding of $R/M \cong k(\beta)$ into V/M sends β to the coset represented by $Y^p + X$. If, instead, one embeds F in $L[X]/(X^2)$ by sending β to $Y^p \in L$ (as in Example 5.11.2), the pullback that is thereby defined *is* valuative.

6. The non-quasilocal, non-integrally closed case

Characterization. The next lemma is useful in showing that if R is valuative but neither integrally closed nor quasilocal, then R has exactly two maximal ideals.

Lemma 6.1. *Let M be a maximal ideal of a domain R such that MR_M is an ideal of a proper overring T of R_M. If $u = b/c$, with $c \in M$ and $b \in R\backslash M$, then $R \subsetneq R[u]$ is not a minimal extension.*

Proof. Clearly, $u^{-1} = c/b \in MR_M$. For each $z \in T$, we then have $zu^{-1} \in MR_M$, and hence $z = um$ for some $m \in MR_M$. Moreover $u \notin T$ (for otherwise, $1 =$

$uu^{-1} \in TMR_M = MR_M$, a contradiction). Hence, $R_M \subsetneq T \subsetneq R_M[u] = R[u]_{R\setminus M}$. Therefore, by [11, Lemme 1.3], $R \subsetneq R[u]$ is not a minimal extension. $\square$

Recall that if $R \subsetneq T$ are domains for which there is a maximal ideal M of R such that $R_M \subsetneq T_M$ is a minimal extension while $R_N = T_N$ for all other maximal ideals N of R, then $R \subsetneq T$ is a minimal extension.

Theorem 6.2. *Let R be a domain that is neither quasilocal nor integrally closed. Then R is valuative if and only if R has* (*exactly*) *two maximal ideals M and N such that M contains each nonmaximal prime ideal of R, R_N is a valuation domain and R_M is a valuative pseudo-valuation domain.*

Proof. Assume R is valuative. By Corollary 3.5, R' is a Bézout domain with at most three maximal ideals; and by Proposition 4.1, $R \subsetneq R'$ is a pointwise minimal (integral) extension. Then it follows from Theorem 4.5 that $R_P = R'_P$ is a valuation domain for each prime ideal P of R except for the crucial maximal ideal M of the extension; and by Theorem 4.6, M is an ideal of R'. It follows that $R_M \subsetneq R'_M$ and MR_M is an ideal of R'_M.

Let $u = b/c$, with $c \in M$ and $b \in R\setminus M$. By Lemma 6.1, $R \subsetneq R[u]$ is not a minimal extension. As $|\operatorname{Max}(R)| < \infty$, the Prime Avoidance Lemma [17, Theorem 81] allows us to further specify c so that M is the only maximal ideal that contains c. If R has more than two maximal ideals or M does not contain each nonmaximal prime, then b and c can be further chosen so that $\sqrt{(R :_R u^{-1})}$ is contained in at least two prime ideals (either because b is contained in more than one maximal ideal or because b is contained in a nonmaximal prime). In this case, $\sqrt{(R :_R u^{-1})} \notin \operatorname{Max}(R)$, whence $R \subsetneq R[u^{-1}]$ is not minimal, contradicting that R is valuative. Therefore, R has exactly two maximal ideals, say (the above) M and (a newly named) N, R_N is a valuation domain, and M contains each nonmaximal prime. By Proposition 2.4, R_M is valuative, and so it remains only to prove that R_M is a pseudo-valuation domain.

Suppose, on the contrary, that R_M is not a pseudo-valuation domain. By Theorem 5.10, the maximal ideal $M'R'_M$ of R'_M is principal and $MR_M = M'^2R'_M$. Since R has only two maximal ideals, globalization shows that $M' = sR'$ for some element s, and so $MR_M = s^2R'_M$. Pick $n \in N\setminus M$, and consider the elements n/s and s/n. We will show that neither $R \subsetneq R[n/s]$ nor $R \subsetneq R[s/n]$ is a minimal extension, thus contradicting the hypothesis that R is valuative. First, we will show that R'_M is contained in $R_M[n/s]$. If $x \in R'_M$, then $x_1 := x(s/n)^2 \in MR_MR'_M = MR_M = s^2R'_M$, and so $x = x_1(n/s)^2 \in R_M[n/s]$ and we have $R_M \subsetneq R'_M \subsetneq R_M[n/s]$. Hence, by [11, Lemme 1.3], $R \subsetneq R[n/s]$ is not a minimal extension. Next, note that $R_N \subsetneq R_N[s/n]$ and $R_M \subsetneq R_M[s/n] = R_M[s]$ (since n is a unit of R_M). Therefore, by [11, Théorème 2.2], $R \subsetneq R[s/n]$ is not a minimal extension, as it does not have a crucial ideal. This completes the proof of the "only if" assertion.

For the converse, assume that R has two maximal ideals M and N such that R_N is a valuation domain and R_M a valuative pseudo-valuation domain. Then $R' = R'_M \cap R_N$ is a Bézout domain with two maximal ideals [17, Theorem 107], $R_N = R'_N$, and $R_M \subsetneq R'_M$ is a pointwise minimal (integral) extension. Therefore, by Theorem 4.5, $R \subsetneq R'$ is a pointwise minimal extension. Using the condition that M contains each nonmaximal prime of R, we will next show that R is valuative.

Let $0 \neq w \in K$. If either w or w^{-1} is in R', we can conclude that at least one of the extensions $R \subseteq R[w]$ and $R \subseteq R[w^{-1}]$ has no proper intermediate ring, since $R \subsetneq R'$ is a pointwise minimal extension. Thus we may assume that neither w nor w^{-1} is in R'. As both R'_M and R_N are valuation domains and $R' = R'_M \cap R_N$, we may further assume that $w \in R'_M \backslash R_N$ and $w^{-1} \in R_N \backslash R'_M$. Since $w^{-1} \notin R'_M$, it follows from Theorem 4.6 that $w \in MR'_M = MR_M$, and so $R_M = R_M[w]$. On the other hand, each proper overring of the valuation domain R_N is a localization of the form R_P, for a suitable nonmaximal prime ideal P of R (contained in N). As M contains each nonmaximal prime ideal of R, we have $R_M \subsetneq R_P$ for each nonmaximal prime P. Since $w \in R_M$, it follows that w belongs to every proper overring of R_N. Hence $R_N \subsetneq R_N[w]$ is a minimal extension. By [10, Proposition 2.1], it follows that $R \subsetneq R[w]$ is a minimal extension. This completes the proof that R is valuative. □

Corollary 6.3. *Let R be a valuative domain that is neither quasilocal nor integrally closed. Then R' is a valuative Bézout domain with exactly two maximal ideals, $M := (R : R')$ and N'. Moreover, $N := N' \cap R$ is the only prime ideal of R that is not a prime ideal of R' and $R_N = R'_{N'}$.*

Proof. By Corollary 3.10, R' is a valuative Bézout domain. Combining Theorem 6.2 with Corollary 3.4, we see that R and R' each have exactly two maximal ideals.

By Proposition 4.1, $R \subsetneq R'$ is a pointwise minimal (integral) extension; and so by Theorem 4.6, the crucial maximal ideal M of this extension is the conductor $M = (R : R')$. Let N be the other maximal ideal of R, and let N' be the maximal ideal of R' that lies over N. As $R'_{N'}$ is an overring of the valuation domain R_N with $N' \cap R = N$, we have $R_N = R'_{N'}$. Also, as M is the conductor of R' into R, N cannot be an ideal of R'. (For an alternate explanation, see [1, Lemma 3.2], bearing in mind that R is not quasilocal.)

By [17, Theorem 107], the maximal ideal of $R' = R'_M \cap R_N$ that lies over M is obtained by intersecting R' with the maximal ideal of R'_M. As R_M is a pseudo-valuation domain whose associated valuation domain is R'_M, this intersection is just $R' \cap MR_M$, and hence is the ideal M itself, since we have

$$M \subseteq R' \cap MR_M = R'_M \cap MR_M \cap R_N = MR_M \cap R_N \subsetneq R_M \cap R_N = R.$$

Next, consider a nonmaximal prime ideal Q of R. As $Q \subseteq M$, the order-isomorphism in Corollary 3.4 gives $\mathfrak{q} \in \mathrm{Spec}(R')$ such that $\mathfrak{q} \subseteq M$ and $\mathfrak{q} \cap R = Q$. Necessarily, $\mathfrak{q} \subsetneq R$, whence $Q = \mathfrak{q} \in \mathrm{Spec}(R')$. □

Remark 6.4. Under the hypothesis that R has two maximal ideals M and N such that R_N is a valuation domain and R_M a valuative pseudo-valuation domain, we have seen in the proof of Theorem 6.2 that $R \subsetneq R'$ is a pointwise minimal extension and R' is a valuative Bézout domain with two maximal ideals. If, in addition, M contains each nonmaximal prime of R, then R is valuative, but if N (not M) contains each nonmaximal prime of R, then R is not valuative.

Overrings. We will prove in Corollary 6.6 that Proposition 5.12 essentially provides the only type of valuative domain with an overring that is not valuative. First, in the spirit of Corollaries 5.4 and 5.8, we describe the overrings of a valuative domain R that is neither quasilocal nor integrally closed. By Theorem 6.2 and Corollary 6.3, R has exactly two maximal ideals M and N, the nonmaximal primes of R are linearly ordered, and each of them is contained in M. Also, R_M is a valuative pseudo-valuation domain and R' is a Bézout domain with two maximal ideals M and N', where N' lies over N and $R_N = R'_{N'}$.

Proposition 6.5. *Let R be a valuative domain that is neither quasilocal nor integrally closed, and let M and N be its maximal ideals, with M the common maximal* *ideal of R and R'. Then each overring of R is valuative and has at most two maximal ideals. Moreover, if S is an overring of R, then it is of one of the following four non-overlapping types.*

- (i) *S is a valuation domain such that either $S = R'_M$ or $S = R_P = R'_P$ for some prime ideal $P \neq M$ of R.*
- (ii) *S is a Bézout domain with two maximal ideals, one of which lies over N, the other one is a (nonzero) prime ideal P of R that is not contained in N, and either $S = R'_M \cap R_N = R'$ (if $P = M$) or $S = R_P \cap R_N$ (if P is a nonmaximal prime). Moreover, $S = (P : P)$.*
- (iii) *S is a pseudo-valuation domain with canonical valuation overring R'_M and $R_M \subseteq S \subsetneq R'_M$.*
- (iv) *S is not integrally closed and has two maximal ideals, M and N_S such that $N_S \cap R = N$. Also $R \subseteq S \subsetneq R'$.*

Proof. By the order-isomorphism between $\mathrm{Spec}(S)$ and the set $\{P \in \mathrm{Spec}(R) \mid PS \neq S\}$ in Corollary 3.4, S has at most two maximal ideals. If S has two maximal ideals, their contractions to R are incomparable and so one of these maximal ideals must lie over N. In any case, by Theorem 6.2 and Corollary 6.3, M is a maximal ideal of both R and R'; also, each nonmaximal prime of R' is a prime of R that is (properly) contained in M. If Q is a nonzero prime of S and $P := Q \cap R$, then $R_P \subseteq S_P \subseteq S_Q$. If $P \neq M$, then R_P is a valuation domain that contains R'. In this case, $S_Q = R_P$ is integrally closed and contains R'.

Suppose S is integrally closed. Then S is an overring of R'. By Corollary 6.3, R' is a valuative Bézout domain, and so is S (by Corollary 3.9). If, in addition, S is quasilocal, then either $S = R'_M$ or $S = R_P$ for some prime ideal P of R other than M; thus, in this case, S is a valuation (hence valuative) domain. On the other hand, suppose S is integrally closed but not quasilocal. Since R' is a Prüfer domain, [14, Theorem 26.1 (2)] gives $S = R'_P \cap R'_{N'}$ where N' is the maximal ideal of R' lying over N and P is a common prime of R' and R that is not contained in N. By [15, Lemma 3], P is a prime ideal of S. To see that P is maximal in S, simply note that $1/t$ is in both R'_P and $R'_{N'} = R_N$ for each $t \in R' \backslash (P \cup N')$ (and use Corollary 3.4). Since $R'_{N'} = R_N$, we have that if $P = M$, then $S = R' = R'_M \bigcap R_N$; if $P \neq M$, we have $S = R_P \cap R_N$. Observe that if Q is a maximal ideal of a domain D, then each maximal ideal of D survives in $(Q : Q)$. Consequently, since S is a Prüfer domain, it follows from [14, Theorem 26.1 (2)] that $S = (P : P)$.

Suppose henceforth that S is not integrally closed. As S_Q is integrally closed for each prime ideal Q of S that does not lie over M, S must have a maximal ideal M_S that lies over M. Then S_{M_S} is an overring of the (valuative) pseudo-valuation domain R_M, and its maximal ideal lies over M. By Corollary 5.4, S_{M_S} is a valuative pseudo-valuation domain with maximal ideal $M_S S_{M_S} = MR_M$. Thus, if S is quasilocal (and not integrally closed), $S = S_{M_S}$ is such that $R_M \subseteq S \subsetneq R'_M$.

Finally, suppose that S is neither quasilocal nor integrally closed, and let N_S denote the maximal ideal of S that lies over N. Then $S_{N_S} = R_N$ is a valuation domain and hence must be $R'_{N'}$. Thus $S = S_{M_S} \cap S_{N_S}$ is properly contained in R' which then implies $M_S = M$ (as $MR'_M \cap R' = M$). That S is valuative follows from Theorem 6.2. □

Corollary 6.6. *Let R be a valuative domain. Then the following conditions are equivalent:*

(1) *Some overring of R is not a valuative domain;*
(2) *R is quasilocal but neither integrally closed nor a pseudo-valuation domain, and R' is not a minimal overring of R.*

Proof. Combine Corollaries 3.9, 5.4 and 5.8 with Propositions 5.12 and 6.5. □

Examples. We end the paper with some examples. It follows from Corollary 6.3 that every non-quasilocal non-integrally closed valuative domain R is obtained as a pullback $R = F \times_L D$ from a valuative Bézout domain $D = R'$ with two maximal ideals $M = M_1$, M_2, where M_1 contains each nonmaximal prime, as illustrated by pictures (1) or (2) in Section 3. By Theorem 5.2, the quotient D/M_1 must be a field L having a subfield F $(= R/M)$ such that either L is a minimal (algebraic) field extension of F or L has prime characteristic p with $L \subseteq F^{1/p}$.

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & F \arrow[d] \\
D \arrow[r] & D/M \cong L
\end{tikzcd}$$

Under this construction, R has two maximal ideals, $M = M_1$ and $N = M_2 \cap R$, R_M is a valuative pseudo-valuation domain with associated valuation domain D_M,

$R_N = D_{M_2}$ is a valuation domain, and M contains each nonmaximal prime of R. (Thus, any such pullback R is valuative by Theorem 6.2, and $R' = D$.) In case $L = D/M$ is a minimal extension of $F = R/M$, then $R \subsetneq R'$ is a minimal (integral) extension. Contrary to the quasilocal case (Proposition 4.13), R' need not contain each minimal overring of R, as is shown by the next explicit example, where D is an intersection of two discrete valuation domains.

Example 6.7. Let $\mathbb{C}(X)$ be the field of rational functions over the complex field. Set $D = V \cap W$ be the intersection of the discrete valuation domains $V = \mathbb{C}[X]_{(X)}$ and $W = \mathbb{C}[X]_{(X+1)}$. Then D is a Bézout domain with two maximal (principal) ideals, $M = M_1 = XD$ and $M_2 = (X+1)D$. Take $R = \mathbb{R} + M$ (where $\mathbb{R}$ denotes the real field). Then R is a (Noetherian) one-dimensional valuative domain with maximal ideals M and $N = M_2 \cap R$. Note that $R' = D$ is the (unique) minimal integral overring of R, but $R \subsetneq R_M$ is a (closed) minimal extension. Indeed, localizing $R \subsetneq R_M$ at M produces a trivial extension, while localizing at N gives the extension $R_N \subsetneq \mathbb{C}(X)$, which is minimal since R_N is a discrete valuation domain. It follows that $R \subsetneq R_M$ is a minimal extension.

Finally, we provide a closely related example of a locally valuative domain with two maximal ideals that is not valuative.

Examples 6.8. With $D = \mathbb{C}[X]_{(X)} \cap \mathbb{C}[X]_{(X+1)}$ and $M_1 = XD$ as in the previous example, now set $R := \mathbb{C} + M_1^2$. Then R is a (Noetherian) one dimensional domain with maximal ideals $M = M_1^2$ and $N = M_2 \cap R$. The integral closure of R is D. In addition, R_M is valuative by Proposition 5.7 (its integral closure is the valuation domain $V := \mathbb{C}[X]_{(X)}$), R_N is the valuation domain $W = \mathbb{C}[X]_{(X+1)}$, and M trivially contains each nonmaximal prime of R. However, since R_M is not a pseudo-valuation domain, it follows from Theorem 6.2 that R is not valuative. A specific "bad pair" is $u = X/(X+1)$, $u^{-1} = (X+1)/X$. In this case, since $R_N \subsetneq \mathbb{C}(X) = R_N[u]$ and $R_M \subsetneq V = R_M[u]$, we see that $R \subsetneq R[u]$ is not a minimal extension (since it does not have a crucial ideal). Moreover, since $R_M \subsetneq V \subsetneq R_M[u^{-1}] = \mathbb{C}(X)$, we see that $R \subsetneq R[u^{-1}]$ is not a minimal extension.

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