Valuative Heights and Infinite Nagata Rings

Ahmed Ayache, Paul-Jean Cahen* and Othman Echi

Résumé. On montre que l'anneau de Nagata $R(\infty)$ à une infinité d'indéterminées sur un anneau R localement de dimension valuative finie est S-fort universel. On étudie aussi la caténarité de $R(\infty)$.

Abstract. We prove here that if the ring R has locally finite valuative dimension then the Nagata ring with infinite indeterminates $R(\infty)$ is a stably strong S-ring. We also study the catenarity of $R(\infty)$.

AMS Subject classification (1991): Primary: 13C05, 13F05, 13F20; Secondary: 13B24, 13G05, 13B22, 13B30, 13F30.

Key words: Nagata rings, polynomials, catenarity, Jaffard rings, strong S-rings.

Introduction

Throughout this paper R is a commutative ring with a unit element. We denote by $R[n]$ the ring of polynomials in n indeterminates on R (but rather by $R[X]$ the ring in one indeterminate), by $\dim R$ the Krull dimension of R and by $\dim_v R$ its valuative dimension, i.e. the limit of the sequence $(\dim R[n]-n)$ and we emphasize that R need not be a domain with such a definition. If $\mathfrak{p}$ is a prime ideal of R, we denote by $\mathrm{ht}\mathfrak{p}$ the height of $\mathfrak{p}$ and by $\mathfrak{p}[n]$ the extension of $\mathfrak{p}$ in $R[n]$ (i.e. the set of polynomials with coefficients in $\mathfrak{p}$). As in [5] we let the *valuative height* of $\mathfrak{p}$, denoted by $\mathrm{ht}_v\mathfrak{p}$, be the valuative dimension of the localization $R_{\mathfrak{p}}$. Letting S be the multiplicative set in $R[n]$ formed by the polynomials whose coefficients generate R, we recall that the localization $R(n) = S^{-1}R[n]$ is called the *Nagata ring* on R with n indeterminates on R. The extension $\mathfrak{p}[n]$ of every prime $\mathfrak{p}$ of R lifts in $R(n)$ and we denote by $\mathfrak{p}(n)$ the corresponding extension $\mathfrak{p}(n) = S^{-1}\mathfrak{p}[n]$ of $\mathfrak{p}$. We

*Supported by the C.N.R.S. (U.R.A. 225)

let $R(0)$ be the ring R and $R(\infty)$ the union $R(\infty) = \bigcup_n R(n)$, we say that $R(\infty)$ is the *infinite Nagata ring* on R. If $\mathfrak{P}$ is a prime ideal of $R[n]$, $R(n)$ or $R(\infty)$ and $\mathfrak{p} = \mathfrak{P} \cap R$, we say that $\mathfrak{P}$ is *above* $\mathfrak{p}$.

The first section is mostly devoted to easy (and sometimes well-known) considerations on the spectrum of $R(n)$ and $R(\infty)$. Recalling that the Krull dimension of $R(\infty)$ is the valuative dimension of R, as proved for a domain by D.E. Dobbs, M. Fontana and S. Kabbaj in [12, corollary 2.5], we prove here similarly that, for any prime $\mathfrak{p}$ of R, the valuative height of $\mathfrak{p}$ is the height of $\mathfrak{p}(\infty)$. We also establish that any prime of finite height in $R(\infty)$ is the extension of a prime of $R(n)$, for n large enough, and that it is an extension of a prime of R if and only if R is quasi-prüferian (as introduced in [6]).

The second section is devoted to various properties of the rings $R(n)$. Recall that a finite dimensional ring R is said to be *Jaffard* if $\dim R[n] = \dim R + n$, for all n [2], or equivalently $\dim R = \dim_{\mathrm{v}} R$, *residually* Jaffard if the quotient of R by any prime $\mathfrak{p}$ is Jaffard, *locally* Jaffard if the localization of R at any prime $\mathfrak{p}$ is Jaffard and lastly *totally* Jaffard if any quotient of any localization (equivalently any localization of any quotient) is Jaffard [11]. We may note that these last two definitions make sense if R is only supposed to be locally finite dimensional. Recall also that R is said to be a strong S-ring if, for any pair $\mathfrak{p} \subset \mathfrak{q}$ of consecutive primes in R, then $\mathfrak{p}[X] \subset \mathfrak{q}[X]$ are consecutive in $R[X]$. If R is a strong S-ring, $R[X]$ need not be so [18]; a ring R such that $R[n]$ is a strong S-ring for any n is said to be a stably strong S-ring. A stably strong S-ring is totally Jaffard and totally Jaffard rings are strong S-rings [11, introduction]. We first characterize these various properties in terms of Nagata rings. Then we prove that, if $s = \dim_{\mathrm{v}} R$ is finite, then $R(n)$ is locally Jaffard for $n \geq s - 1$, and above all that, if $R(\infty)$ is locally finite dimensional, then it is a stably strong S-ring.

In the third and last section we define the ring R to be *valuatively catenarian* or simply *v-catenarian* if $R(\infty)$ is locally finite dimensional and catenarian. We show that a v-catenarian strong S-ring is totally Jaffard and catenarian. We give however an example of a totally Jaffard catenarian ring which is not v-catenarian and also of a v-catenarian ring which is not catenarian. We show also that $R(1)$ may be catenarian whereas $R(2)$ is not. Many examples use gluing techniques with which we assume the reader to be familiar.

Terminology is standard as in [13]; we use "$\subset$" to denote proper containment. Transcendence degrees play an important role in the examples; if $A \subseteq B$ are two domains we denote by d.t.$[A : B]$ the transcendence degree of the field of fractions of B over the field of fractions of A.

1 Valuative heights and quasi-prüferians rings

Recall first that, for any integers n and m, $R(m + n) = R(m)(n)$, and for any prime $\mathfrak{p}$ of R, $\mathfrak{p}(m+n) = \mathfrak{p}(m)(n)$ [1, lemma p.138]; similarly, for any n, $R(\infty) = R(n)(\infty)$, and $\mathfrak{p}(\infty) = \mathfrak{p}(n)(\infty)$. Recall also that a prime ideal of $R[n]$ lifts in $R(n)$ if and only if it is contained in the extension $\mathfrak{m}[n]$ of some maximal ideal $\mathfrak{m}$ of R ; in particular the maximal ideals of $R(n)$ are the extensions of maximal ideals of R and $R(n)$ is quasi-local (with maximal ideal $\mathfrak{m}(n)$) if and only if R is quasi-local (with maximal ideal $\mathfrak{m}$). It may be worth noticing that, contrarily to polynomial rings, $(R(n))_{\mathfrak{p}}$ is in general distinct from $R_{\mathfrak{p}}(n)$. Indeed any prime above $\mathfrak{p}$ in $R(n)$ lifts in $(R(n))_{\mathfrak{p}}$ whereas $\mathfrak{p}(n)$ is the only prime above $\mathfrak{p}$ which lifts in $R_{\mathfrak{p}}(n)$. In fact it is clear that $R_{\mathfrak{p}}(n)$ is the localization of $R(n)$ with respect to the prime ideal $\mathfrak{p}(n)$, hence a localization of $(R(n))_{\mathfrak{p}}$. As already shown by D.E. Dobbs at al in the particular case of a domain in [12, corollary 2.5], $\dim R(\infty) = Sup_n \dim R(n) = \dim_{\mathrm{v}} R$, since $\dim R(n) = \dim R[n] - n$ [2, proposition 1.21]. Similarly, for any prime $\mathfrak{p}$ of R, $\mathrm{ht}_{\mathrm{v}}\mathfrak{p} = \dim_{\mathrm{v}} Rp = \dim R_{\mathfrak{p}}(\infty)$. But $\dim R_{\mathfrak{p}}(\infty) = \mathrm{ht}\mathfrak{p}(\infty)$, since $R_{\mathfrak{p}}(\infty)$ is the localization of $R(\infty)$ with respect to the prime $\mathfrak{p}(\infty)$. We may summarize these equalities in the following proposition:

Proposition 1.1 *(i)* $\dim_{\mathrm{v}} R = \dim R(\infty)$,

(ii) for any prime $\mathfrak{p}$ *of* R, $\mathrm{ht}_{\mathrm{v}}\mathfrak{p} = \mathrm{ht}\mathfrak{p}(\infty)$.

This proposition makes sense even if the dimensions (or heights) involved are infinite (for example $dimR(\infty)$ is infinite if and only if $dim_v R$ is infinite).

Our next result is analogous to the special chain theorem [9, Theorem 1] according which, if $\mathfrak{P}$ is a prime of $R[n]$ above $\mathfrak{p}$, $\mathrm{ht}\mathfrak{P} = \mathrm{ht}\mathfrak{p}[n] + ht(\mathfrak{P}/\mathfrak{p}[n])$ (the formulation of this theorem is often different but equivalent):

Proposition 1.2 *(i) let* $\mathfrak{P}$ *be a prime of* $R(n)$ *above* $\mathfrak{p}$*, then*

$$\mathrm{ht}\mathfrak{P} = \mathrm{ht}\mathfrak{p}(n) + ht(\mathfrak{P}/\mathfrak{p}(n)),$$

(ii) let $\mathfrak{P}$ *be a prime of* $R(\infty)$ *above* $\mathfrak{p}$*, then*

$$\mathrm{ht}\mathfrak{P} = Sup_n ht(\mathfrak{P}\cap R(n)) = \mathrm{ht}\mathfrak{p}(\infty)+\mathrm{ht}(\mathfrak{P}/\mathfrak{p}(\infty)) = \mathrm{ht}_{\mathrm{v}}\mathfrak{p}+\mathrm{ht}(\mathfrak{P}/\mathfrak{p}(\infty)).$$

Proof. The first assertion is immediate from the special chain theorem, since $R(n)$ is a localization of $R[n]$. For the second, letting $\mathfrak{P}_n$ be the intersection $\mathfrak{P}_n = \mathfrak{P} \cap R(n)$, we then have $\mathrm{ht}\mathfrak{P} = Sup_n \mathrm{ht}\mathfrak{P}_n$, since $R(\infty)$ is the union of the rings $R(n)$ and any chain of primes of $R(n)$ lifts in $R(\infty)$ (taking the extension of each prime of the chain) [12, lemma 2.1], [7, VIII, Exercise 11, p.82]. Similarly $\mathrm{ht}\mathfrak{p}(\infty) = Sup_n \mathrm{ht}\mathfrak{p}(n)$ and $\mathrm{ht}(\mathfrak{P}/\mathfrak{p}(\infty)) = Sup_n \mathrm{ht}(\mathfrak{P}_n/\mathfrak{p}(n))$ $\diamond$

Maximal ideals of $R(\infty)$ being extensions of ideals of R, we derive easily that the valuative height of any prime of R is finite if and only if $R(\infty)$ is locally finite dimensional; under these conditions we say that R is *locally v-finite dimensional*; this is for instance clearly the case if R is Noetherian. The main result of this section is that, if $R(\infty)$ is locally finite dimensional, then every prime ideal of $R(\infty)$ is the extension of a prime ideal of $R(n)$, for n large. Since any prime of $R(\infty)$ is then the localization of a prime of *finite height* of $R[\infty]$ (the ring of polynomials in infinitely many indeterminates), this is a direct consequence of the following:

Theorem 1.3 *Let* $\mathfrak{P}$ *be a prime of finite height in* $R[\infty]$ *and* $\mathfrak{P}_n$ *be the intersection* $\mathfrak{P}_n = \mathfrak{P} \cap R[n]$*. Then there exists an integer* k *such that, for* $n \geq k$*,*

(i) $\mathfrak{P}_{n+1}$ *is the extension of* $\mathfrak{P}_n$ *to* $R[n+1]$*,*

(ii) $\mathfrak{P}$ *is the extension of* $\mathfrak{P}_n$ *to* $R[\infty]$*.*

Proof. For the first assertion, If the extension $\mathfrak{P}_n[1]$ of $\mathfrak{P}_n$ to $R[n+1] = R[n][1]$ is such that $\mathfrak{P}_n[1] \subset \mathfrak{P}_{n+1}$, then $\mathrm{ht}\mathfrak{P}_{n+1} > \mathrm{ht}\mathfrak{P}_n$, since any chain of $R[n]$ lifts in $R[n+1]$ (taking the extension of each prime of the chain). If the set of integers such that $\mathfrak{P}_n[1] \subset \mathfrak{P}_{n+1}$ were infinite, so would be $\mathrm{ht}\mathfrak{P}$, contrary to the hypothesis. The second assertion follows immediately, since $\mathfrak{P} = \bigcup_n \mathfrak{P}_n$ $\diamond$

The integer k of the previous theorem depends on the prime $\mathfrak{P}$, for instance $k = 0$ if and only if $\mathfrak{P}$ is the extension of a prime ideal of R.

According to the next example, this integer may be arbitrarily large, even if the valuative dimension of R is finite:

Example 1.4 Let $R = F[X, Y]$, where F is a field. R is a Noetherian unique factorization domain of valuative dimension 2. In the ring $R[n] = R[T_1, \ldots, T_n]$, the polynomial $f = XTn - Y$ is irreducible, hence the principal ideal (f) is prime (note that this ideal is contained in the extension $\mathfrak{m}[n]$ of the maximal ideal $\mathfrak{m} = (X, Y)$, hence lifts as a non-zero prime ideal of $R(n)$). Letting $\mathfrak{P}$ be the extension of (f) to $R[\infty]$, then $\mathfrak{P}_n = (f)$ whereas $\mathfrak{P}_{n-1} = (0)$, since $(f) \cap R[n-1] = (0)$. In conclusion $\mathfrak{P}_n$ is not the extension of $\mathfrak{P}_{n-1}$.

Many properties of the ring R transfer well to the Nagata rings $R(n)$ and $R(\infty)$. For instance if R is Noetherian we derive easily from the previous theorem that the primes of finite height in $R[\infty]$ are finitely generated, thus that all the primes of $R(\infty)$ are finitely generated and in conclusion that $R(\infty)$ is itself Noetherian, as in [14, theorem 6]. Along this line, recalling that a prime ideal of $R[n]$ lifts in $R(n)$ if and only if it is contained in the extension $\mathfrak{m}[n]$ of some maximal ideal $\mathfrak{m}$ of R, and that a *quasi-prüferian* ring is a ring such that any prime ideal of $R[n]$ contained in an extension of a prime ideal of R is itself an extension [6], we also get easily the following:

Proposition 1.5 *The following assertions are equivalent:*

(i) R is quasi-prüferian,

(ii) for any n, $R(n)$ is quasi-prüferian,

(iii) there exists an integer n such that $R(n)$ is quasi-prüferian,

(iv) $R(X)$ is quasi-prüferian,

(v) any prime ideal of $R(\infty)$ is the extension of a prime ideal of R,

(vi) $R(\infty)$ is quasi-prüferian.

Proof. To say that R is quasi-prüferian means that any prime ideal of $R(n)$ (for any n, some n ore equivalently for $n = 1$) is the extension of a prime of R [6, proposition 1.5], but then, for every integer m, any prime ideal of $R(n)(m) = R(n + m)$ is an extension of a prime of $R(n)$ and thus $R(n)$ is quasi-prüferian. Hence (i) implies (ii), (iii) and (iv), which are equivalent.

Supposing (ii), let $\mathfrak{P}$ be a prime ideal of $R(\infty)$ above $\mathfrak{p}$; then, for any n, $\mathfrak{P}\cap R(n) = \mathfrak{p}(n)$, thus $\mathfrak{p} = \mathfrak{p}(\infty)$, since $\mathfrak{P} = \bigcup_n(\mathfrak{P}\cap R(n))$, hence (ii) implies (v). Since $R(\infty)(m)$ is isomorphic to $R(\infty)$, (v) implies that any prime of $R(\infty)(m)$ is the extension of a prime of $R(\infty)$ thus of a prime of R, and hence that $R(\infty)$ is quasi-prüferian. Lastly, supposing that $R(n)$ is quasi-prüferian (where n is an integer or $n = \infty$), let $\mathfrak{P} \subset \mathfrak{q}[X]$ in $R[X]$ (where $\mathfrak{q}$ is a prime of R), then $\mathfrak{P}(n) \subset \mathfrak{q}(n)[X]$ in $R(n)[X]$, hence $\mathfrak{P}(n)$ is the extension $\mathfrak{p}'[X]$, of a prime ideal $\mathfrak{p}'$ of $R(n)$ and therefore $\mathfrak{P} = R[X]\cap\mathfrak{p}'[X] = (R\cap\mathfrak{p}')[X]$ is the extension of the prime $\mathfrak{p} = R\cap\mathfrak{p}'$, and R is quasi-prüferian [6, proposition 1.5] $\diamond$

A quasi-prüferian domain being Prüfer if and only if it is integrally closed, we recover immediately that R is a Prüfer domain if and only if this is the case of $R(X)$, $R(n)$ or $R(\infty)$ [6, theorem 33.4].

In fact the Nagata rings turn out to have some properties that the ground ring do not have to start with, for instance if R is a Prüfer domain, the Nagata rings are Bezout domains [1, theorem 2] (whereas a Prüfer domains need not always be Bezout). In the next section we establish such properties of the Nagata rings.

2 Jaffard rings and strong S-rings

We first characterize Jaffard properties in terms of Nagata rings. Next result is easily derived from S. Kabbaj's [16, Lemma 1.4]; considering a pair $\mathfrak{p}\subset\mathfrak{q}$ of primes of R, we let $\mathfrak{q}/\mathfrak{p}$ be the prime ideal of $R/\mathfrak{p}$, quotient of $\mathfrak{q}$ by $\mathfrak{p}$; we note that $R/\mathfrak{p}(\infty)$ is isomorphic to $R(\infty)/\mathfrak{p}(\infty)$ and that $(\mathfrak{q}/\mathfrak{p})(\infty)$ corresponds to $\mathfrak{q}(\infty)/\mathfrak{p}(\infty)$ under this isomorphism.

Proposition 2.1 *Let R be locally finite dimensional,*

(i) R is Jaffard if and only if $\dim R = \dim R(\infty)$,

(ii) R is locally Jaffard if and only if, for any prime $\mathfrak{p}$ of R, $\mathrm{ht}\mathfrak{p} = \mathrm{ht_v}\mathfrak{p} = \mathrm{ht}\mathfrak{p}(\infty)$,

(iii) R is totally Jaffard if and only if, for any pair $\mathfrak{p}\subset\mathfrak{q}$ of primes of R, $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = \mathrm{ht_v}(\mathfrak{q}/\mathfrak{p}) = \mathrm{ht}(\mathfrak{q}(\infty)/\mathfrak{p}(\infty))$.

We can also characterize strong S-ring in terms of Nagata rings:

Proposition 2.2 *The ring R is a strong S-ring if and only if, for any pair $\mathfrak{p} \subset \mathfrak{q}$ of consecutive primes of R, any one of the following assertions holds:*

(i) $\mathfrak{p}(1) \subset \mathfrak{q}(1)$ *are consecutive in* $R(1)$,

(ii) there exists an integer $n \geq 1$ *such that* $\mathfrak{p}(n) \subset \mathfrak{q}(n)$ *are consecutive in* $R(n)$,

(iii) for all $n \geq 1$, $\mathfrak{p}(n) \subset \mathfrak{q}(n)$ *are consecutive in* $R(n)$,

(iv) $\mathfrak{p}(\infty) \subset \mathfrak{q}(\infty)$ *are consecutive in* $R(\infty)$.

Proof. Localizing at $\mathfrak{q}$ and taking the quotient by $\mathfrak{p}$ we may assume that R is a dimension 1 quasi-local domain, with maximal ideal $\mathfrak{q}$ and that $\mathfrak{p} = (0)$. Therefore the result follows easily from the fact that a one dimensional ring is a strong S-ring if and only if it is Jaffard (equivalently $\dim R(n) = \dim R(\infty) = 1$ or $\mathrm{ht}\mathfrak{q}(n) = \mathrm{ht}\mathfrak{q}(\infty) = 1$) [2, theorem 1.10] $\diamond$

In a previous paper we had shown that, if the valuative dimension of R is finite, then $R[n]$ is locally Jaffard for $n \geq \dim_{\mathrm{v}} R - 1$ [11, proposition 1]. Since $R(n)$ is a localization of $R[n]$, our next result is an immediate consequence:

Proposition 2.3 *Let n be such that* $n \geq \dim_{\mathrm{v}} R - 1$, *then $R(n)$ is locally Jaffard.*

We show next that $R(n)$ is totally Jaffard for n large if and only if it is a strong S-ring. First we set a slight improvement of S. Kabbaj's [17, theorem 2], according which, if $R(n)$ is a strong S-domain for all n, then R is Jaffard.

Lemma 2.4 *Let R be locally v-finite dimensional such that one of the following hypotheses is satisfied:*

(i) $R(n)$ *is a strong S-ring for all* n,

(ii) $\dim_{\mathrm{v}} R = s$ *and* $R, R(1), \ldots, R(s-2)$ *are strong S-rings,*

(iii) $\dim_{\mathrm{v}} R = s$ *and* $R[n]$ *is a strong S-ring for some* $n \geq s - 2$,

then R is totally Jaffard.

Proof. For any pair of primes $\mathfrak{p} \subset \mathfrak{q}$ of R, we want to show that $T = R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ is Jaffard, i.e. that $\dim T(n+1) = \dim T(n)$, for all n. If $R(n)$ is a strong S-ring, so is $T(n)$, because strong S-rings are stable under quotient and localization. Thus $\dim T(n)[1] = \dim T(n) + 1$, hence $\dim T(n+1) = \dim T(n)[1] - 1 = \dim T(n)$. This proves (i). If $\dim_{\mathrm{v}} R = s$, then $\dim_{\mathrm{v}} T \leq s$, hence $T(n)$ is Jaffard, for $n \geq s-1$ [proposition2.3], therefore $\dim T(n+1) = \dim T(n)$, for $n \geq s-1$; this proves (ii). Lastly, if $R[n]$ is a strong S-ring, so is $R[m]$, for $m \leq n$, and so is $R(m)$, since strong S-rings are stable under localization $\diamond$

We derive immediately the following:

Proposition 2.5 *Let R be locally v-finite dimensional and k be an integer. The following assertions are equivalent:*

(i) $R(n)$ is a strong S-ring for $n \geq k$,

(ii) $R(n)$ is totally Jaffard for $n \geq k$.

Question 2.6 *Is $R(n)$ totally Jaffard (or equivalently a strong S-ring) for $n \geq \dim_{\mathrm{v}} R - 1$, or at least for n large?*

The next proposition is a partial answer:

Proposition 2.7 *If $dim_v R = s \leq 2$, then, for $n \geq s-1$, $R(n)$ is totally Jaffard.*

Proof. For any prime $\mathfrak{p}$ of R, $\dim_{\mathrm{v}} R/\mathfrak{p} \leq 2$. For $n \geq s-1$, $(R/\mathfrak{p})(n)$ is then a locally Jaffard domain [proposition 2.3] such that $\dim(R/\mathfrak{p})(n) \leq 2$, hence it is totally Jaffard [11, corollaire 1, p.128]. So is $R(n)/\mathfrak{P}$, for any prime $\mathfrak{P}$ of $R(n)$, since $\mathfrak{P}$ is above some $\mathfrak{p}$ of R, hence $R(n)/\mathfrak{P}$ is a quotient of $(R/\mathfrak{p})(n)$ $\diamond$

From propositions 2.5 and 2.7, we derive immediately the following:

Corollary 2.8 *If $dim R \leq 2$, the following assertions are equivalent:*

(i) R is totally Jaffard,

(ii) $R(n)$ is a strong S-ring for all n.

Remarks 2.9 (i) If $\dim R = 0$,the assertions of the previous corollary are always satisfied. If $\dim R = 1$, one may easily conclude from [2, theorem 1.10] that they are also equivalent to (iii) "R is Jaffard" and (iv) "R is a stably strong S-ring". If $\dim R = 2$ and R is a domain, these assertions are satisfied if and only if R is locally Jaffard; if in addition R is quasi-local they are even satisfied if and only if R is Jaffard, i.e. $\dim_v R = 2$ [11, proposition 2 & corollaire 1].
(ii) If $\dim R = d$, $d \geq 2$, R may be a strong S-ring but not Jaffard [11, example 3]. If $d \geq 3$, R may be totally Jaffard but not a strong S-ring [11, example 8].
(iii) If R is a stably strong S-ring then $R(n)$ is clearly a strong S-ring for all n. The converse does not hold: [5, example 5.3] is a dimension 2, quasi-local and totally Jaffard domain which is not a stably strong S-ring; according to the previous corollary, $R(n)$ is a strong S-ring for all n.
(iv) The ring R is a stably strong S-ring if and only if $R[n]$ is totally Jaffard for all n; similarly it results from [proposition 2.5] that $R(n)$ is a strong S-ring for all n if and only if $R(n)$ is totally Jaffard for all n.
(v) Contrary to polynomials, it may be that $R(n)$ is a strong S-ring but $R(m)$ is not for $m < n$. Indeed, if $\dim R = 1$ and $\dim_v R = 2$, then R is not a strong S-ring [2, theorem 1.10], but, for $n \geq 1$, $R(n)$ is a strong S-ring [proposition 2.7].

It is an open question whether $R[X, Y]$ is a strong S-ring when this is the case for $R[X]$. Similarly we ask:

Question 2.10 *If $R(1)$ is a strong S-ring, is $R(2)$ also a strong S-ring?*

D.E. Dobbs et al. have shown that the Nagata ring $R(\infty)$ over a finite dimensional Jaffard domain R is itself a Jaffard domain [12, corollary 2.5]. We close this section with a significant improvement: if $R(\infty)$ is only supposed to be locally finite dimensional, then it is a stably strong S-ring. Note however that $R[\infty]$ is not in general a strong S-ring: A. Bouvier et al. [8, proposition 2.1] gave an example where $R[\infty]$ has an homomorphic image which is not an S-domain; we may however emphasize that the kernel of this homomorphism is a prime of *infinite* height, whereas for prime ideals of *finite* height, we first prove the following:

Lemma 2.11 *Let $\mathfrak{P} \subset \mathfrak{Q}$ be consecutive primes of finite height in $R[\infty]$; then $\mathfrak{P}[1] \subset \mathfrak{Q}[1]$ are consecutive in $R[\infty][1]$.*

Proof. From [theorem 1.3], there is an integer k such that $\mathfrak{P}$ and $\mathfrak{Q}$ are extensions of primes of $R[k]$. Replacing R by $R[k]$, since $R[\infty]$ and $R[\infty][k]$ are clearly isomorphic, we may thus consider that $\mathfrak{P} = \mathfrak{p}(\infty)$ and $\mathfrak{Q} = \mathfrak{q}[\infty]$, where $\mathfrak{P}$ and $\mathfrak{Q}$ are respectively above the primes $\mathfrak{p}$ and $\mathfrak{q}$ of R. The infinite polynomial ring $R[\infty]$ is the set theoretic union of the rings $R[n]$ and $R[\infty][1]$ the set theoretic union of the rings $R[n][1]$; thus $R[\infty][1]$ is isomorphic to $R[\infty]$ since $R[n][1]$ is isomorphic to $R[n+1]$. Similarly $\mathfrak{P} = \mathfrak{p}(\infty)$ and $\mathfrak{Q} = \mathfrak{q}[\infty]$ are respectively the union of the primes $\mathfrak{p}[n]$ and $\mathfrak{q}[n]$, whereas $\mathfrak{P}[1]$ and $\mathfrak{Q}[1]$ are respectively the union of the primes $\mathfrak{p}[n][1]$ and $\mathfrak{q}[n][1]$, thus $\mathfrak{P}[1]$ and $\mathfrak{Q}[1]$ correspond to the primes $\mathfrak{P}$ and $\mathfrak{Q}$ under the isomorphism of $R[\infty][1]$ with $R[\infty]$ $\diamond$

Since $R(\infty)[m]$ is a localisation of $R[\infty][m]$, which is isomorphic to $R[\infty]$, two consecutive primes of $R(\infty)[m]$ correspond to consecutive primes of finite height in $R[\infty]$; thus we get:

Theorem 2.12 *If R is locally v-finite dimensional, then $R(\infty)$ is a stably strong S-ring.*

Corollary 2.13 *If R is locally v-finite dimensional, then $R(\infty)$ is totally Jaffard.*

3 Valuative catenarity

If $\mathfrak{P}$ is a prime of $R(\infty)$ above $\mathfrak{p}$ then it contains $\mathfrak{p}(\infty)$ and is contained in the extension $\mathfrak{m}(\infty)$ of a maximal ideal $\mathfrak{m}$, hence a locally v-finite dimensional ring R is catenarian if and only if, for any pair of primes $\mathfrak{p} \subset \mathfrak{q}$ of R, all the saturated chains of primes between $\mathfrak{p}(\infty)$ and $\mathfrak{q}(\infty)$ have same length in $R(\infty)$; under such conditions we say that R is *v-catenarian.*

We note that this property is clearly stable under quotient and localization. It also follows immediately from the fact that $R(n+m)$ is isomorphic to $R(n)(m)$ and $R(n)(\infty)$ to $R(\infty)$ that R is v-catenarian if and only if $R(n)$ is v-catenarian for some integer n or equivalently fo all n.

Next proposition links the catenarity and the v-catenarity of R:

Proposition 3.1 *Let R be a locally finite dimensional strong S-ring.*

(i) If $R(n)$ is catenarian for some integer n, then R is catenarian.

(ii) If R is v-catenarian then R is catenarian and totally Jaffard.

Proof. Let $\mathfrak{p} \subset \mathfrak{q}$ be a pair of prime ideals of R, and $\mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_h = \mathfrak{q}$ a saturated chain of primes between them. The length of this chain is h. Letting n be an integer or $n = \infty$, the chain $\mathfrak{p}(n) \subset \mathfrak{p}_1(n) \subset \ldots \subset \mathfrak{p}_h(n) = \mathfrak{q}(n)$ is saturated in $R(n)$, since R is a strong S-ring [proposition 2.2]. And since $R(n)$ is supposed to be catenarian, h is therefore the length of any saturated chain between $p(n)$ and $q(n)$, thus R is catenarian. In the case of $n = \infty$, we get $h = \text{ht}(\mathfrak{q}(\infty)/\mathfrak{p}(\infty)) = \text{ht}_{\text{v}}(\mathfrak{q}/\mathfrak{p}) = \text{ht}(\mathfrak{q}/\mathfrak{p})$, thus R is totally Jaffard [proposition 2.1] $\diamond$

According to the next example, a catenarian strong S-ring need not be v-catenarian

Example 3.2 Let R be a dimension 2 strong S-domain which is not Jaffard [11, example 3], R is catenarian but not v-catenarian [proposition3.1(ii)]. Moreover R (which can be taken to be quasi-local) is such that $R(1)$ is catenarian (because it is dimension 2) whereas, for $n \geq 2$, $R(n)$ is not (indeed, if $\mathfrak{m}$ is the maximal ideal of R and $(0) \subset \mathfrak{p} \subset \mathfrak{m}$ is a saturated chain in R, then $(0) \subset \mathfrak{p}(n) \subset \mathfrak{m}(n)$ is a saturated chain in $R(n)$ [proposition2.2] whereas the height of $\mathfrak{m}(n)$ is greater than 2). However it is worth noticing that there are no known examples such that $R[1]$ is catenarian and $R[2]$ is not.

We now give more examples of domains which are catenarian and not v-catenarian or conversely, more precisely of a catenarian *and totally Jaffard domain* which is not v-catenarian and of a *locally Jaffard* v-catenarian domain which is not catenarian. We note that in both cases we do need the dimension to be at least 3 (if R is Jaffard and of lesser dimension, then R and $R(\infty)$ are always catenarian). The last example provides also a locally Jaffard v-catenarian domain which is not a strong S-ring [proposition3.1] whereas a locally Jaffard catenarian domain is always totally Jaffard [11, proposition 2 & corollaire 2] (hence a strong S-ring).

Both examples use gluing techniques as in [11]: we let K be a field and B a K-algebra which is a locally Jaffard semi-local domain with two maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively of height m and n (this algebra may be a Noetherian domain [11, example A, p.134]). Then we let $I = \mathfrak{m} \cap \mathfrak{n}$ and $R = K + I$. Thus R is a quasi-local domain, with maximal ideal I and $dimR = Sup(m, n)$. There are two saturated chains of primes between (0) and I, one is of length m, the other of length n, hence according to the values given to these two integers, R is or is not catenarian.

Example 3.3 A dimension 3 catenarian and totally Jaffard domain, which is not v-catenarian.

We let $m = 1$, $n = 3$, d.t.$[B/\mathfrak{m} : K] = 1$ and d.t.$[B/\mathfrak{n} : K] = 0$.
In this case R is catenarian, dimension 3 and locally Jaffard [11, corollary 2, p.131], thus totally Jaffard [11, corollary 1, p.128]. However we prove $R(n)$ not to be catenarian, for $n \geq 1$ or $n = \infty$:
We consider an element α such that $\alpha \in \mathfrak{n}$ and the class $\overline{\alpha}$ of α is transcendent over K (this is possible because $\mathfrak{m} + \mathfrak{n} = B$). We then consider the prime ideal $\mathfrak{P}_\alpha = (\alpha T_1 - 1) \cap R[n]$ in $R[n]$. It is an height 1 prime contained in $I[n]$ [10, proof of lemma 6]. We claim that $\mathfrak{P}_\alpha \subset I[n]$ are consecutive in $R[n]$. Indeed, since $\mathfrak{P}_\alpha$ does not contain I, there is only one prime lifting it in $B[n]$ [10, proposition 0] and this is $\mathfrak{P}'_\alpha = (\alpha T_1 - 1) \cap B[n]$. Thus, if there were a chain $(0) \subset \mathfrak{P}_\alpha \subset \mathfrak{Q} \subset I[n]$, this chain would lift in $B[n]$ as $(0) \subset \mathfrak{P}'_\alpha \subset \mathfrak{Q}' \subset \mathfrak{M}$ [10, proposition 4]. Therefore ht$\mathfrak{M} \geq 3$ and $\mathfrak{M}$ would contain the polynomial $f = \alpha T_1 - 1$, thus could not contain $\mathfrak{n}$ and would then be above $\mathfrak{m}$. Both primes $\mathfrak{m}[n]$ and $\mathfrak{M}$ would be above the prime $I[n]$ of $R[n]$; however d.t.$[(B[n]/\mathfrak{m}[n]) : (R[n]/I[n])] = 1$, thus ht$(\mathfrak{M}/\mathfrak{m}[n]) = 1$ [10, lemma 7]. On the other hand ht$\mathfrak{m}[n] = 1$ and this implies ht$\mathfrak{M} =$ ht$\mathfrak{m}[n] +$ ht$(\mathfrak{M}/\mathfrak{m}[n]) = 2$, from the special chain theorem [9, theorem 1], a contradiction.

Example 3.4 A dimension 3 v-catenarian and locally Jaffard domain which is not catenarian.
We let $m = 2$, $n = 3$, d.t.$[B/\mathfrak{m} : K] = 1$ and d.t.$[B/\mathfrak{n} : K] = 0$. We suppose moreover that B is universally catenarian (this is certainly the case if it is the localization of a finite K-algebra).
In this case R is not catenarian, dimension 3 and locally Jaffard [11, corollary 2, p.131]. We prove now that R is v-catenarian:
Since the longest chains in $R(\infty)$ are length 3 and necessarily between (0) and $I(\infty)$, it is clear that R could only fail to be v-catenarian, if there were a saturated chain of length two between them, which would come (by localization) from a saturated chain $(0) \subset \mathfrak{P} \subset I[\infty]$ in $R[\infty]$. The prime $\mathfrak{P}$ would be the extension of some prime $\mathfrak{P}_n$ of $R[n]$ to $R[\infty]$ [theorem 1.3] and clearly the chain $(0) \subset \mathfrak{P}_n \subset I[n]$ would then be saturated in $R[n]$. This last chain would lift in $B[n]$ as $(0) \subset \mathfrak{Q} \subset \mathfrak{M}$ [10, proposition 4]. Taking $\mathfrak{M}$ to be minimal among the primes lifting $I[n]$, the chain $(0) \subset \mathfrak{Q} \subset \mathfrak{M}$ would be saturated, hence $\mathfrak{M}$ would be height 2 (since B is supposed to be universally catenarian). Necessarily $\mathfrak{M}$ would be above the prime $\mathfrak{m}$ of B (since $\mathfrak{n}$ is height 3) and would be the extension $\mathfrak{M} = \mathfrak{m}[n]$ (since $\mathfrak{m}$ is height 2). The rings $R[n] \subset B[n]$ share the ideal $I[n]$, $\mathfrak{m}[n]$ is above $I[n]$ and d.t.$[(B[n]/\mathfrak{m}[n]) : (R[n]/I[n])] = 1$, therefore there would be a chain of the

type $(0) \subset \mathfrak{P}_n[1] \subset \mathfrak{P}' \subset I[n+1]$, in the ring $R[n][1] = R[n+1]$ [10, lemma 6]. Taking the extensions of each prime to $R[\infty]$, we would lastly get the chain $(0) \subset \mathfrak{P} \subset \mathfrak{P}'[\infty] \subset I[\infty]$ and this would bring a contradiction.

Remark 3.5 In the particular case where R is quasi-prüferian, it is clear however that the spectra of R, $R(X)$, $R(n)$ and $R(\infty)$ are order isomorphic, thus R is catenarian if and only if $R(n)$ is catenarian for all n or equivalently, for some n, $n = 1$ or $n = \infty$.

We finally compare the catenarity of $R(\infty)$ and $R(n)$ for n large:

Proposition 3.6 *If R is locally v-finite dimensional and if there is an integer k such that $R(n)$ is catenarian for $n \geq k$, then R is v-catenarian.*

Proof. To prove that R is v-catenarian, we may consider its quotients by prime ideals hence assume it is a domain. We thus want to show that, for any pair $\mathfrak{P} \subset \mathfrak{Q}$ of consecutive primes of $R(\infty)$, we have $\mathrm{ht}\mathfrak{Q} = \mathrm{ht}\mathfrak{P} + 1$. From [theorem1.3], there exists an integer n such that $\mathfrak{P}$ and $\mathfrak{Q}$ are extensions of primes $\mathfrak{P}_n$ and $\mathfrak{Q}_n$ of $R(n)$. Taking $n \geq k$, $R(n)$ is a catenarian domain, thus $\mathrm{ht}\mathfrak{Q}_n = \mathrm{ht}\mathfrak{P}_n + 1$, since $\mathfrak{P}_n$ and $\mathfrak{Q}_n$ are clearly consecutive. If $\mathfrak{Q}$ is above $\mathfrak{q}$, we may localize R at $\mathfrak{q}$, hence assume that $\dim_{\mathrm{v}} R = s$ is finite. Taking $n \geq s - 1$, $R(n)$ is locally Jaffard [proposition 2.3] and the result follows from the equalities $\mathrm{ht}\mathfrak{P}_n = \mathrm{ht}\mathfrak{P}_n(\infty) = \mathrm{ht}\mathfrak{P}$ and $\mathrm{ht}\mathfrak{Q}_n = \mathrm{ht}\mathfrak{Q}_n(\infty) = \mathrm{ht}\mathfrak{Q}$ [proposition 2.1] $\diamond$

In connection with question 2.6 we ask if the converse holds:

Question 3.7 *Assuming R is v-catenarian, is there an integer k such that $R(n)$ is catenarian for $n \geq k$?*

Remark 3.8 (i) The answer would be positive if it were for question 2.6. Indeed, there would be an integer k such that $R(n)$ is a strong S-ring for $n \geq k$, so if R were v-catenarian, $R(n)$ would immediately be also v-catenarian and thus would be catenarian for $n \geq k$ [proposition 3.1].
(ii) Conversely, if the answer to question 3.8 were positive and R were a v-catenarian ring such that $\dim_{\mathrm{v}} R = s$ is finite, there would be an integer k such that $R(n)$ is a strong S-ring for $n \geq k$. Indeed, considering the quotient of R by a prime ideal, hence assuming R is a domain, there would be an integer k such that $R(n)$ is catenarian for $n \geq k$; on the other hand $R(n)$ is locally Jaffard, for $n \geq ks - 1$ [proposition 2.3] and a locally Jaffard

catenarian domain is totally Jaffard hence a strong S-ring [11, corollaire 1, p.128].
(iii) It may be that $R(n)$ is catenarian, for all n, whereas R is not a strong S-ring. For instance, if $\dim R = 1$ and $\dim_{\mathrm{v}} R = 2$ then R is not a strong S-ring [2, theorem 1.10] whereas $\dim R(n) \leq 2$ hence $R(n)$ is catenarian. We note however that, for polynomials, if $R[X]$ is catenarian then R is a strong S-ring [8, lemma 2.3]. Thus $R(n)$ may be catenarian, for all n, whereas R is not universally catenarian.
(iv) If R is a noetherian domain of dimension 2 such that $R[X]$ is not catenarian, as in a famous example by Nagata [19, p.205], then R is a *strong S-ring* such that $R(n)$ is catenarian, for all n, but which is not universally catenarian.

References

[1] D.D.Anderson. Some remarks on the rings $R(X)$. *Comment. Math. Univ. St Pauli.* **26**, (1977) 137-140.

[2] D.F.Anderson, A.Bouvier, D.E.Dobbs, M.Fontana and S.Kabbaj. On Jaffard domains. *Expo. Math.* **5**, (1988) 145-175.

[3] J.T.Arnold and R.Gilmer. The dimension sequence of a commutative ring. *Amer. J. of Math.* **96 (3)**, (1974) 385-408.

[4] A.Ayache. About a conjecture on Nagata rings. *(to appear).*

[5] A.Ayache and P.-J.Cahen. Anneaux vérifiant absolument l'inégalité ou la formule de la dimension. *Bolletino U.M.I.***7 (6-B)**,(1992) 39-65.

[6] A.Ayache, P.-J.Cahen and O.Echi. Anneaux Quasi-Prüfériens et P-anneaux. *Prépublication de l'URA 225.* (1993) 93-07.

[7] N.Bourbaki. *Algèbre commutative, chapitres 8-9*, Masson, Paris, (1983).

[8] A.Bouvier,D.E.Dobbs and M.Fontana. Universally catenarian integral domains. *Advances in Math.* **72**, (1988) 211-238.

[9] J.W.Brewer,P.A.Montgomerry,P.A.Rutter and W.J.Heinzer. Krull dimension of polynomial rings. *LNM.* **311**, 26-46, Berlin-Heidelberg-New York (1973).

[10] P.-J.Cahen. Couple d'anneaux partageant un idéal. *Archiv der Math.* **51**, (1988) 505-514.

[11] P.-J.Cahen. Construction B,I,D et anneaux localement ou résiduellement de Jaffard. *Archiv der Math.* **54**, (1990) 125-141.

[12] D.E.Dobbs, M.Fontana and S.Kabbaj. Direct limits of Jaffard domains and S-domains. *Comment. Math. Univ. St Pauli.* **39**, (1990) 143-155.

[13] R.Gilmer. *Multiplicative ideal theory.* Marcel Dekker, New-york, 1974.

[14] R.Gilmer and W.J. Heinzer. The Noetherian property for quotient rings of infinite polynomial rings. *Proc. Amer. Math. Soc.* **76**, (1979) 1-7.

[15] P.Jaffard. *Théorie de la dimension dans les anneaux de polynômes.* Gauthiers-Villars, Paris (1960).

[16] S.Kabbaj. La formule de la dimension pour les S-domaines forts universels. *Bolletino U.M.I.* **6 (5)**, (1986) 145-161.

[17] S.Kabbaj. Une conjecture sur les anneaux de Nagata. *J. of Pure and Applied Algebra.* **64**, (1990) 263-268.

[18] S.Malik and J.L.Mott. Strong S-domains. *J. of Pure and Applied Algebra.* **28**, (1983) 249-264.

[19] M.Nagata. *Local rings.* Interscience, New York (1962).

Ahmed AYACHE
Poste de l'ancienne Université
B.P. 12460 SANAA, République du Yemen

Paul-Jean CAHEN
Service 322 Faculté des sciences de Saint-Jérome Université d'Aix-Marseille III.
13397 Marseille cedex 20, France
E-mail: cahen @ frmop11.cnusc.fr

Othman ECHI
Département de Mathématiques
Faculté des Sciences de Sfax B.P. W, 3038 Sfax, Tunisie